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The Role of the Weyl Projective Curvature Tensor and Its Relation to other Curvatures Tensors in Spacetime Geometry

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Abstract

This research delves into the intricate realm of Finsler geometry, with a particular focus on curvature tensors. The paper aims to establish novel identities that govern the expansion of these tensors within the broader context of Finsler space. Through rigorous mathematical analysis, we explore the inter relationships between various curvature tensors and their corresponding expansions. The derived identities not only deepen our understanding of the intrinsic geometric properties of Finsler spaces but also offer potential applications in fields such as physics and engineering where Finsler geometry plays a significant role. This work contributes to the ongoing development of Finsler geometry and provides a foundation for future research in this area. The expansion curvature tensor W_{ijk}^h is an important geometric object in Finsler spaces. It measures the deviation of the geodesic flow from a parallel flow. In this paper, we investigate some identities for the expansion curvature tensor W_{ijk}^h in Finsler spaces. These identities provide valuable insights into the geometric properties of Finsler spaces and can be used to derive new results in Finsler geometry. We investigate some identities between Weyl Curvature Tensor W_{ijk}^h and some other curvature tensors.

Keywords: *Finsler space, Berwald covariant derivative expansion, curvature tensor, identities, geometric properties..*

1. Introduction

Finsler geometry, an extension of Riemannian geometry, provides a robust framework for investigating spaces with anisotropic characteristics. This paper explores the complexities of Finsler spaces, concentrating on the identities governing the expansion of curvature tensors. By analyzing these identities, we seek to illuminate the geometric and algebraic structures inherent in Finsler geometry. The examination of curvature tensors in Finsler spaces is crucial due to their significance in characterizing the intrinsic curvature of these spaces. These tensors embody information concerning the deviation of geodesics and the parallel transport of vectors. By scrutinizing the expansion identities for curvature tensors, we aspire to uncover profound connections among the diverse curvature invariants and to acquire a more comprehensive c of the curvature properties of Finsler spaces.

Furthermore, the findings presented in this paper hold promise for applications across diverse domains, including physics, engineering, and computer science. For example, Finsler geometry has been utilized in the formulation of relativistic gravity theories and in the creation of innovative materials exhibiting anisotropic characteristics. Curvature tensors play a pivotal role in differential geometry. Notable examples of curvature tensors include the Riemannian curvature tensor, Weyl projective curvature tensor, M-projective curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, and P_1-Curvature tensor. The Riemannian curvature tensor was introduced by Bernhard Riemann in 1854 in his Habilitation vortrag "Ueber die Hypothesen, welche der Geometric zuGrundeliegen." The conformal curvature tensor is another significant curvature tensor with numerous applications in differential geometry.

In this paper we investigate some identities between Weyl curvature tensor W_{ijk}^h and some others curvature tensors by using Berwald covariant derivative. We first introduce the basic concepts and the relationship between these curvature tensors and Weyl curvature tensor W_{ijk}^h, we also introduce the basic concepts of Berwald covariant derivative and investigate some identities between some important curvature tensors and Weyl curvature tensor. Then we derive expansion for Berwald covariant derivative in general form. Finally, we apply this expansion and identities some relationships between some curvature tensors and Weyl curvature tensor by given some examples. The concept of the three-dimensional of Riemannian space with recurrent curvature was studied and explored by Rund [18]. In the context of recurrent Finsler spaces, the analysis of generalized curvature tensors relies on the Berwald curvature tensor, which has been discussed by Abdallah [1], AL-Qashbari[6], and others. Properties of the curvature tensor W_{jkh}^i were investigated by Ahsan& Ali [4], Hadi[10], Al-Qashbari and Qasem[5], Abu-Donia, Shenawy and Abdelhameed[2], and others. Ahsan[23],Ahsan and Ali [22] studied on some properties of W-curvature tensor, Chagpar, Pokhariyal and Moindi [9] introduced P_1-Curvature tensor, Ali and Salman [13] studied some properties of M-projective curvature tensor. Using Berwald's connection, the covariant derivative B_k of a general tensor T_j^i along the x^k direction is

$$B_k T_j^i = \partial_k T_j^i - (\dot{\partial}_r T_j^i) G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r . \quad (1.1)$$

According to reference [7], there exists a relationship between the quantities g_{ij} and g^{ij}

(a) $g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 \;, & if \quad i = k \\ 0 \;, & if \quad i \neq k \end{cases}$ and (b) $g_{ij}\, y^i = y_j$. (1.2)

Moreover, the covariant derivative of $\mathcal{B}_m$ is given by

(a) $\mathcal{B}_m \delta_h^k R_{ij} = \lambda_m \delta_h^k\, R_{ij}$, (b) $\mathcal{B}_m g_{ij}\, R_h^k = \lambda_m g_{ij}\, R_h^k$, (1.3)

(a) $\mathcal{B}_m R \delta_k^h g_{ij} = \lambda_m R \delta_k^h g_{ij}$ and (b) $\mathcal{B}_m R R_{ij} = \lambda_m R R_{ij}$. (1.4)

A substantial number of scholars have put forth the following identities in their studies (see [17, 14, 7, 21]).

$(a)\ \overline{W}_{lijk} g^{lh} = \overline{W}_{ijk}^h, (b)\ R_{lijk} g^{lh} = R_{ijk}^h, \ \ (c)\ R_{lj} g^{lh} = R_j^h,$ (1.5)

(a) $\delta_k^i y^k = y^i$ and (b) $\delta_j^k y_k = y_j$. (1.6)

Also from [16, 13, 7, 20] we have

$$(a) y_j y^j = F^2 \ and \ (b)\ R_i^i = R. \qquad (1.7).$$

The outline of this paper is as follows:

Following the introductory section and the preliminaries section, we explore the expansion of any curvature tensor in terms of the Berwald covariant derivative. In section2, establish the relationships between the Weyl curvature tensor and other curvature tensors. Subsection3.1 delves into the expansion of the Berwald covariant derivative for an arbitrary curvature tensor. Finally, in subsection3.2, we analyze the identities introduced in section2 using the previously described expansion.

2. Preliminaries

In Finsler geometry, there exists a mathematical relationship between any two curvature tensors. This paper will explore the specific relationship between the Weyl curvature tensor and the following curvature tensors:

2.1.Weyl Projective Curvature Tensor W_{ijk}^h

The Weyl projective curvature tensor serves as a geometric tool for describing the curvature of a spacetime or, more broadly, a pseudo-Riemannian manifold. While closely linked to the Riemann curvature tensor, it exhibits invariance under conformal transformations, remaining unchanged even when the manifold's metric is scaled by a non-zero function. This property renders the Weyl projective curvature tensor invaluable for studying the geometry of spacetime, especially in scenarios where the exact spacetime metric remains uncertain.

Furthermore, the Weyl projective curvature tensor maintains a close relationship with the Cotton tensor. The Cotton tensor quantifies the shear of curvature, vanishing if and only if the spacetime possesses conformal flatness. Consequently, the Weyl projective curvature tensor vanishes if and only if the spacetime exhibits local isometry to flat spacetime.

Definition2.1. The Riemannian curvature tensor in terms of Weyl projective curvature tensor W_{ijk}^h is defined as Musavvir Ali, Naeem Ahmad and Mohammad Salman (2022) and Zafar and Musavvir (2013).

$$W_{jkh}^i = R_{jkh}^i - \frac{1}{n-1}\left(\delta_h^i R_{jk} - \delta_k^i R_{jh}\right) \ . \qquad (2.1)$$

In the 4-dimensional vector space V over the field F, we have

$$W_{jkh}^i = R_{jkh}^i - \frac{1}{3}\left(\delta_h^i R_{jk} - \delta_k^i R_{jh}\right). \quad (2.2)$$

The tensors $W_{jkh}^i, W_{jk}^i, R_{jkh}^i$ and R_{jk}^i satisfy the following identities

a) $W_{jkh}^i\, y^j = W_{kh}^i$, b) $W_{jk}^i\, y^j = W_k^i$, c) $R_{jkh}^i\, y^j = R_{kh}^i$, d) $R_{jk}^i\, y^j = R_k^i$.(2.3)

Also, if we suppose that the tensor W_j^i and W satisfy the following identities

a) $W_{ki}^i = W_k$ and b) $W_i^i = W$. (2.4)

2.2. Concircular Curvature Tensor M_{ijk}^h

The concircular curvature tensor, a geometric entity introduced within the field of differential geometry, serves as a quantifier of the curvature inherent in spacetime or, more broadly, pseudo-Riemannian manifolds. It maintains a close association with the conformal curvature tensor (alternately referred to as the Weyl curvature tensor) and the projective curvature tensor. Notably, the concircular curvature tensor vanishes exclusively when the manifold exhibits concircular flatness.

Definition2.2. The concircular curvature tensor $\mathrm{M_{hijk}}$ in a 4-dimensional spacetime is defined as in Ahsan and Siddiqui (2009).

$$M_{hijk} = R_{hijk} - \frac{1}{12} R\left(g_{ij} g_{hk} - g_{ik} g_{hj}\right).$$
(2.5)

Also

$$M_{ijk}^{h}=R_{ijk}^{h}-\frac{1}{12}R\left(g_{ij}\delta_k^h-g_{ik}\delta_j^h\right).$$

(2.6)

Combining equations (2.2) and (2.6), we obtain

$$M_{ijk}^{h}=W_{ijk}^{h}+\frac{1}{3}\left(\delta_k^hR_{ij}-\delta_j^hR_{ik}\right)-\frac{1}{12}R\left(g_{ij}\delta_k^h-g_{ik}\delta_j^h\right).$$

(2.7)

2.3. $\mathbf{P_1}$-Curvature Tensor

The P1-curvature tensor, a geometric entity within the realm of differential geometry, serves as a quantifier of the curvature inherent in spacetime or, more broadly, pseudo-Riemannian manifolds. Its close association with the Ricci curvature tensor and scalar curvature is noteworthy. The P1-curvature tensor vanishes exclusively when the manifold exhibits Ricci flatness and possesses constant scalar curvature. The tensor has been defined by Pokhariyal (1973).

$$P_1(X,Y,Z,T)=R(X,Y,Z,T)+\frac{1}{2(n-1)}[g(Y,Z)Ric(X,T)-g(Y,T)Ric(X,Z)-g(X,Z)Ric(Y,T)+g(X,T)Ric(Y,Z).$$

(2.8)

We consider the P_1-curvature tensor in the index notation as Chagpar, Pokhariyal and Moindi (2021)

$$P_{1_{hijk}}=R_{hijk}+\frac{1}{2(n-1)}\left[g_{ij}R_{hk}-g_{ik}R_{hj}-g_{hj}R_{ik}+g_{hk}R_{ij}\right].$$

(2.9)

This can be written as

$$P_{1_{ijk}}^{h}=R_{ijk}^{h}+\frac{1}{2(n-1)}\left[g_{ij}R_k^h-g_{ik}R_j^h-\delta_j^hR_{ik}+\delta_k^hR_{ij}\right].$$

(2.10)

In the 4-dimensional vector space V over the field F, we have

$$P_{1_{ijk}}^{h}=R_{ijk}^{h}+\frac{1}{6}\left[g_{ij}R_k^h-g_{ik}R_j^h-\delta_j^hR_{ik}+\delta_k^hR_{ij}\right].$$

(2.11)

If we put (2.2) and (2.11) together, we get

$$P_{1_{ijk}}^{h}=W_{ijk}^{h}+\frac{1}{3}\left(\delta_k^hR_{ij}-\delta_j^hR_{ik}\right)+\frac{1}{6}\left[g_{ij}R_k^h-g_{ik}R_j^h-\delta_j^hR_{ik}+\delta_k^hR_{ij}\right]$$

. (2.12)

2.4. Conformal Curvature Tensor C_{ijk}^{h}

The conformal curvature tensor, alternatively identified as the Weyl curvature tensor, represents a geometric entity introduced within the realm of differential geometry. Serving as a quantifier of curvature within spacetime or, more expansively, pseudo-Riemannian manifolds, it mirrors the tidal force experienced by a body traversing a geodesic path, akin to the Riemann curvature tensor. However, the Weyl tensor differentiates itself from the Riemann curvature tensor by excluding information regarding volumetric changes, focusing exclusively on the distortion of the body's shape as influenced by the tidal force.

Definition2.3. Zafar and Musavvir (2013) express the conformal curvature tensor C_{ijk}^{h} as follows:

$$C_{ijk}^{h}=R_{ijk}^{h}-\frac{1}{2}\left(\delta_j^hR_{ik}-\delta_k^hR_{ij}+R_j^hg_{ik}-R_k^hg_{ij}\right)-\frac{1}{6}R\left(g_{ij}\delta_k^h-g_{ik}\delta_j^h\right).$$

(2.13)

From (2.2) and (2.13), we can see that

$$C_{ijk}^{h}=W_{ijk}^{h}+\frac{1}{3}\left(\delta_k^hR_{ij}-\delta_j^hR_{ik}\right)-\frac{1}{2}\left(\delta_j^hR_{ik}-\delta_k^hR_{ij}+R_j^hg_{ik}-R_k^hg_{ij}\right)-\frac{1}{6}R\left(g_{ij}\delta_k^h-g_{ik}\delta_j^h\right).$$

(2.14)

2.5. Conharmonic Curvature Tensor L_{ijk}^{h}

The conharmonic curvature tensor, a geometric entity within the realm of differential geometry, serves as a generalization of both the projective curvature tensor and the conformal curvature tensor. Its properties have been extensively investigated across diverse contexts, encompassing Riemannian geometry, Kähler geometry, and cosmological studies.

Definition2.4. For V_4 the Conharmonic curvature tensor L_{ijk}^{h} defined as Ishii (1957) and Siddiqui and Ahsan (2010)

$$L_{ijk}^{h}=R_{ijk}^{h}-\frac{1}{2}\left(g_{ij}R_k^h+\delta_k^hR_{ij}-\delta_j^hR_{ik}-g_{ik}R_j^h\right).$$

(2.15)

Using (2.2) and (2.15), we find that

$$L_{ijk}^{h}=W_{ijk}^{h}+\frac{1}{3}\left(\delta_k^hR_{ij}-\delta_j^hR_{ik}\right)-\frac{1}{2}\left(g_{ij}R_k^h+\delta_k^hR_{ij}-\delta_j^hR_{ik}-g_{ik}R_j^h\right).$$

(2.16)

Where: W_{ijk}^{h} is the Weyl projective curvature tensor and R_k^h is the torsion tensor.

2.6. Projective Curvature Tensor $\overline{W}_{ijk}^{h}$

The projective curvature tensor $\overline{W}$ is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and the conharmonic curvature tensor. It has been studied in a variety of contexts, including Riemannian geometry, Kähler geometry, and cosmology. The properties of an M-projective curvature tensor were proposed by Pokhariyal and Mishra in (1971). This tensor is described as follows:

The M-projective curvature tensor is a tensor field of type (1,3) on a manifold M. It is defined by the following equation:

$$\bar{W}(X,Y,Z,T)=\bar{R}(X,Y,Z,T)-\frac{1}{2(n-1)}[S(Y,Z)g(X,T)-S(X,Z)g(Y,T)$$

$$+g(Y,Z)S(X,T)-g(X,T)S(Y,Z)]. \quad (2.17)$$

Where: $\bar{W}(X,Y,Z,T)=g(W(X,Y)Z,T)$ and $\bar{R}(X,Y,Z,T)=g(R(X,Y)Z,T)$.

R is the Riemann curvature tensor, S is the Ricci tensor, g is the metric tensor, n is the dimension of the manifold.

The $\bar{W}$-projective curvature tensor has a number of interesting properties. For example, it is invariant under conformal transformations. This means that it is the same for two metrics that are conformally equivalent. The $\bar{W}$-projective curvature tensor also vanishes if and only if the manifold is Ricci-flat.

The $\bar{W}$-projective curvature tensor has been used to study a variety of geometric problems. For example, it has been used to classify Riemannian manifolds, to study the geometry of Kähler manifolds, and to develop new models of gravity.

The local coordinates expression of equation (2.17) as follows

$$\bar{W}_{lijk}=R_{lijk}-\frac{1}{2(n-1)}[R_{ij}g_{lk}-R_{lj}g_{ik}+g_{ij}R_{lk}-g_{lj}R_{ik}]. \quad (2.18)$$

Assuming $n=4$ in equation (2.18) and contracting with g^{lh} byusing (1.2a), (1.5a), (1.5b) and (1.5c) the $\bar{W}$-projective curvature tensor is given by

$$\bar{W}^h_{ijk}=R^h_{ijk}-\frac{1}{6}(\delta^h_k R_{ij}-\delta^h_j R_{ik}+g_{ij}R^h_k-g_{ik}R^h_j). \quad (2.19)$$

The combination of equations (2.2) and (2.19) leads to the conclusion that

$$\bar{W}^h_{ijk}=W^h_{ijk}+\frac{1}{3}(\delta^h_k R_{ij}-\delta^h_j R_{ik})-\frac{1}{6}(\delta^h_k R_{ij}-\delta^h_j R_{ik}+g_{ij}R^h_k-g_{ik}R^h_j). \quad (2.20)$$

3. Main Results.

3.1. Expansion Curvatures Tensors in Finsler Space

The expansion curvature tensor, closely linked to both the Riemann curvature tensor and the Berwald curvature tensor, vanishes exclusively when the underlying Finsler manifold exhibits flatness. Within the realm of Finsler geometry, the expansion curvature tensor T emerges as a geometric entity, serving as a quantifier of the curvature inherent in Finsler manifolds, which stand as generalizations of Riemannian manifolds. In our previous work, Al-Qashbari and Al-Maisary (2024), we introduced the generalized Berwald covariant derivative β_m for any tensor T^h_{ijk}. In section3 of last our paper AL-Qashbari, and Halboup (2024), we introduced theorem3.1 and we investigated the following identity

$$\beta_m T^i_{jkh}=\lambda_m T^i_{jkh}+\mu_m(\delta^i_h g_{jk}-\delta^i_k g_{jh})+\frac{1}{4}[R^i_h g_{jk}-R^i_k g_{jh}]. \quad (3.1)$$

Also in section4 in last our paper AL-Qashbari, and Halboup (2024), we introduced theorem3.2.1 and we investigated the following identity

$$\beta_m W^i_{jkh}=\lambda_m W^i_{jkh}+\mu_m(\delta^i_h g_{jk}-\delta^i_k g_{jh})+\frac{1}{4}[R^i_h g_{jk}-R^i_k g_{jh}]. \quad (3.2)$$

From the previous steps, we can conclude the following theorem

Theorem3.1.1. The expansion of (1.1) is given by (3.1).

The dimensionality of many curvatures tensors operators will be extended in accordance with theorem 3.1.1.

Definition3.1.1. A Finsler space of tensor W^i_{jkh} is called as projective curvature tensor and is known as satisfies (3.2), will be called a generalized-first recurrent space. We shall call this Finsler space as a generalized $\mathcal{B}W$-first-recurrent space and we denoted by $G\mathcal{B}W\text{-}FRF_n$.

Transvecting condition to a higher dimensional space (3.2) by y^j, using (2.3a) and (1.2b), we get

$$\beta_m W^i_{kh}=\lambda_m W^i_{kh}+\mu_m(\delta^i_h y_k-\delta^i_k y_h)+\frac{1}{4}[R^i_h y_k-R^i_k y_h]. \quad (3.3)$$

Again, transvecting condition to a higher dimensional space (3.3) by y^k, using (2.3b), (1.7a) and (1.6a), we get

$$\beta_m W^i_h=\lambda_m W^i_h+\mu_m(\delta^i_h F^2-y_h y^i)+\frac{1}{4}[R^i_h F^2-R^i_k y_h y^k]. \quad (3.4)$$

Therefore, the proof of theorem is completed, we can say

Theorem3.1.2. In $G\mathcal{B}W\text{-}FRF_n$, covariant derivative for Berwald of first order for torsion tensor W^i_{kh} and deviation tensor W^i_h are given by (3.3) and (3.4) respectively.

Contracting the index space by summing over i and h in the conditions (3.3) and (3.4), using (2.4a), (2.4b), (1.2a), (1.6b), (1.7a) and (1.7b), we get

$$\beta_m W_k=\lambda_m W_k+(n-1)\mu_m y_k+\frac{1}{4}[R\,y_k-R^i_k y_i]. \quad (3.5)$$

And

$$\beta_m W = \lambda_m W + (n-1)\mu_m F^2 + \frac{1}{4}[RF^2 - R^i_k\, y_i y^k]. \quad (3.6)$$

From conditions (3.5) and (3.6), we show that the curvature vector W_k and the curvature scalar W cannot equal to zero because if the vanishing of any one of these would imply $\lambda_m = 0$ and $\mu_m = 0$, that is a contradiction.

So the proof of theorem is completed, we can say

Theorem 3.1.3. In GBW-FRF$_n$, the vector W_k and the scalar W in equations (3.5) and (3.6), are non-vanishing, respectively.

4.2. Study the Application of Identities in Expression

Mathematical identities are equations that hold true universally, irrespective of the specific values assigned to their variables. These identities serve as valuable tools for simplifying expressions, solving equations, and establishing theorems. Our focus was on investigating the expansion of the Berwald covariant derivative for any curvature tensor as presented in equation (3.2).

$$\beta_m W^h_{ijk} = \lambda_m W^h_{ijk} + \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{1}{4}[R^h_k g_{ij} - R^h_j g_{ik}]. \quad (3.7)$$

We suppose that (3.7) holds to investigate the following identities

3.2.1. By tack away Berwald covariant derivative for (2.2), we have

$$\beta_m R^h_{ijk} = \beta_m W^h_{ijk} + \frac{1}{3}\beta_m(\delta^h_k R_{ij} - \delta^h_j R_{ik}). \quad (3.8)$$

Using(1.3a) and (3.7) in(3.8), we get

$$\beta_m R^h_{ijk} = \lambda_m W^h_{ijk} + \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{1}{4}[R^h_k g_{ij} - R^h_j g_{ik}] + \frac{1}{3}\lambda_m(\delta^h_k R_{ij} - \delta^h_j R_{ik}).$$

This gives

$$\beta_m R^h_{ijk} = \lambda_m \left[W^h_{ijk} + \frac{1}{3}(\delta^h_k R_{ij} - \delta^h_j R_{ik})\right] + \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{1}{4}[R^h_k g_{ij} - R^h_j g_{ik}]. \quad (3.9)$$

By using (2.2) in (3.9), we have

$$\beta_m R^h_{ijk} = \lambda_m R^h_{ijk} + \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{1}{4}[R^h_k g_{ij} - R^h_j g_{ik}]. \quad (3.10)$$

From the previous steps, we can conclude the following theorem

Theorem3.2.1: The expansion derivative for the Berwald of Riemann curvature tensor R^h_{ijk} (2.2) satisfies the equation (3.10).

Transvecting condition to a higher dimensional space (3.10) by y^i, using (1.2b) and (2.3c) we get

$$\beta_m R^h_{jk} = \lambda_m R^h_{jk} + \mu_m(\delta^h_k\, y_j - \delta^h_j y_k) + \frac{1}{4}[R^h_k\, y_j - R^h_j y_k]. \quad (3.11)$$

Again, transvecting condition to a higher dimensional space(3.11) by y^j, using (1.7a), (1.6a), and (2.3d), we get

$$\beta_m R^h_k = \lambda_m R^h_k + \mu_m(\delta^h_k\, F^2 - y^h y_k) + \frac{1}{4}[R^h_k\, F^2 - R^h_j\, y^j y_k]. \quad (3.12)$$

Therefore, the proof of theorem is completed, we can say

Theorem3.2.2. In covariant derivative for the Berwald of fourth order for torsion tensor R^i_{kh} and deviation tensor R^i_h are given by (3.11) and (3.12) respectively.

3.2.2. Tack away Berwald covariant derivative for (2.7), we have

$$\beta M^h_{ijk} = \beta\left[W^h_{ijk} + \frac{1}{3}(\delta^h_k R_{ij} - \delta^h_j R_{ik}) - \frac{R}{12}(g_{ij}\delta^h_k - g_{ik}\delta^h_j)\right]. \quad (3.13)$$

Using (1.3a), (1.4a), (3.7) and (3.13), we get

$$\beta M^h_{ijk} = \lambda_m W^h_{ijk} + \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{1}{4}[R^h_k g_{ij} - R^h_j g_{ik}] + \frac{\lambda}{3}(\delta^h_k R_{ij} - \delta^h_j R_{ik}) - \frac{1}{12}\lambda_m[R(g_{ij}\delta^h_k - g_{ik}\delta^h_j)].$$

Or can be written as

$$\beta M^h_{ijk} = \lambda_m\left[W^h_{ijk} + \frac{1}{3}(\delta^h_k R_{ij} - \delta^h_j R_{ik}) - \frac{R}{12}(g_{ij}\delta^h_k - g_{ik}\delta^h_j)\right] + \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{1}{4}[R^h_k g_{ij} - R^h_j g_{ik}]. \quad (3.14)$$

From (2.7) and (3.14), we have

$$\beta M^h_{ijk} = \lambda_m M^h_{ijk} + \mu_m(\delta^h_k g_{ij} - \delta^h_j g_{ik}) + \frac{1}{4}[R^h_k g_{ij} - R^h_j g_{ik}]. \quad (3.15)$$

In conclusion the proof of theorem is completed, we can determine

Theorem3.2.3. The expansion derivative for the Berwald of Concircular curvature tensor M^h_{ijk} (2.7) satisfies the equation (3.15).

3.2.3. Tack away Berwald covariant derivative for (2.12), we have

$$\beta P^h_{1_{ijk}} = \beta W^h_{ijk} + \frac{1}{3}\beta\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) + \frac{1}{6}\beta\left(g_{ij}R^h_k - g_{ik}R^h_j - \delta^h_j R_{ik} + \delta^h_k R_{ij}\right).$$
(3.16)

From(1.3a), (1.3b), (3.7)and(3.16), we get

$$\beta P^h_{1_{ijk}} = \lambda_m W^h_{ijk} + \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \frac{1}{4}\left[R^h_k g_{ij} - R^h_j g_{ik}\right] + \frac{1}{3}\lambda_m\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right)$$

$$+ \frac{1}{6}\lambda_m\left(g_{ij}R^h_k - g_{ik}R^h_j - \delta^h_j R_{ik} + \delta^h_k R_{ij}\right).$$

Or can be written as

$$\beta P^h_{1_{ijk}} = \lambda_m\left[W^h_{ijk} + \frac{1}{3}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) + \frac{1}{6}\left(g_{ij}R^h_k - g_{ik}R^h_j - \delta^h_j R_{ik} + \delta^h_k R_{ij}\right)\right]$$

$$+ \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \frac{1}{4}\left[R^h_k g_{ij} - R^h_j g_{ik}\right].$$

(3.17)

By using (2.12) in (3.17), we have

$$\beta P^h_{1_{ijk}} = \lambda_m P^h_{1_{ijk}} + \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right)\frac{1}{4}\left[R^h_k g_{ij} - R^h_j g_{ik}\right].$$
(3.18)

The proof of theorem is completed, we conclude

Theorem3.2.4. The expansion derivative for the Berwald of P_1-curvature tensor $P^h_{1_{ijk}}$ (2.12) satisfies the equation (3.18).

3.2.4. Tack away Berwald covariant derivative for (2.14), we have

$$\beta C^h_{ijk} = \beta W^h_{ijk} + \frac{\beta}{3}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) + \frac{\beta}{2}\left(\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_j g_{ik} - R^h_k g_{ij}\right)$$

$$+ \frac{\beta R}{6}\left(g_{ij}\delta^h_k - g_{ik}\delta^h_j\right). \quad (3.19)$$

Using(1.3a), (1.3b), (1.4a) and (4.1) in (3.19), we get

$$\beta C^h_{ijk} = \lambda_m W^h_{ijk} + \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \frac{1}{4}\left[R^h_k g_{ij} - R^h_j g_{ik}\right] + \frac{1}{3}\lambda_m\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right)$$

$$+ \frac{1}{2}\lambda_m\left(\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_j g_{ik} - R^h_k g_{ij}\right) + \frac{1}{6}\lambda_m R\left(g_{ij}\delta^h_k - g_{ik}\delta^h_j\right)$$

Or, we can write as

$$\beta C^h_{ijk} = \lambda_m\left[W^h_{ijk} + \frac{1}{3}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) + \frac{1}{2}\left(\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_j g_{ik} - R^h_k g_{ij}\right)\right.$$

$$\left. + \frac{1}{6}R\left(g_{ij}\delta^h_k - g_{ik}\delta^h_j\right)\right] + \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \frac{1}{4}\left[R^h_k g_{ij} - R^h_j g_{ik}\right].$$
(3.20)

By using (2.14) in (3.20), we have

$$\beta C^h_{ijk} = \lambda_m C^h_{ijk} + \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \frac{1}{4}\left[R^h_k g_{ij} - R^h_j g_{ik}\right].$$
(3.21)

In conclusion the proof of theorem is completed, we can determine

Theorem3.2.5. The expansion derivative for the Berwald of Con-formal curvature tensor C^h_{ijk} (2.14) satisfies the equation (3.21).

3.2.5. Tack away Berwald covariant derivative for (2.16), we have

$$\beta L^h_{ijk} = \beta W^h_{ijk} + \frac{1}{3}\beta\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) - \frac{1}{2}\beta\left(g_{ij}R^h_k + \delta^h_k R_{ij} - \delta^h_j R_{ik} - g_{ik}R^h_j\right).$$
(3.22)

Using(1.3a), (1.3b), (3.7) in (3.22), we get

$$\beta L^h_{ijk} = \lambda_m W^h_{ijk} + \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \frac{1}{4}\left[R^h_k g_{ij} - R^h_j g_{ik}\right]$$

$$+ \frac{1}{3}\lambda_m\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) - \frac{1}{2}\lambda_m\left(g_{ij}R^h_k + \delta^h_k R_{ij} - \delta^h_j R_{ik} - g_{ik}R^h_j\right)$$

.

Or can be written as

$$\beta L^h_{ijk} = \lambda_m\left[W^h_{ijk} + \frac{1}{3}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) - \frac{1}{2}\left(g_{ij}R^h_k + \delta^h_k R_{ij} - \delta^h_j R_{ik} - g_{ik}R^h_j\right)\right]$$

$$+ \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \frac{1}{4}\left[R^h_k g_{ij} - R^h_j g_{ik}\right].$$

(3.23)

From (2.16) and (3.23), we get

$$\beta L^h_{ijk} = \lambda_m L^h_{ijk} + \mu_m\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right) + \frac{1}{4}\left[R^h_k g_{ij} - R^h_j g_{ik}\right].$$
(3.24)

Thus, the proof of theorem is completed, we get

Theorem3.2.6. The expansion derivative for the Berwald of Con-harmonic curvature tensor L^h_{ijk} (2.16) satisfies the equation (3.24).

3.2.6. Tack away Berwald covariant derivative for (2.20), we have

$$\beta_m \overline{W}^h_{ijk} = \beta_m W^h_{ijk} + \frac{1}{3}\beta_m\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right)$$

$$-\frac{1}{6}\beta_m\left(\delta_l^h R_{jk}-\delta_k^h R_{jl}+g_{jk}R_l^h-g_{jl}R_k^h\right). \quad (3.25)$$

Using (1.3a), (1.3b) and (3.7) in(3.25), we get

$$\beta_m \overline{W}_{ijk}^h = \lambda_m W_{ijk}^h+\mu_m\left(\delta_k^h g_{ij}-\delta_j^h g_{ik}\right)+\frac{1}{4}\left[R_k^h g_{ij}-R_j^h g_{ik}\right]$$

$$+\frac{1}{3}\lambda_m\left(\delta_k^h R_{ij}-\delta_j^h R_{ik}\right)-\frac{1}{6}\lambda_m\left(\delta_l^h R_{jk}-\delta_k^h R_{jl}+g_{jk}R_l^h-g_{jl}R_k^h\right).$$

This can be written as

$$\beta_m \overline{W}_{ijk}^h = \lambda_m\left[W_{ijk}^h+\frac{1}{3}\left(\delta_k^h R_{ij}-\delta_j^h R_{ik}\right)-\frac{1}{6}\left(\delta_l^h R_{jk}-\delta_k^h R_{jl}+g_{jk}R_l^h-g_{jl}R_k^h\right)\right]$$

$$+\mu_m\left(\delta_k^h g_{ij}-\delta_j^h g_{ik}\right)+\frac{1}{4}\left[R_k^h g_{ij}-R_j^h g_{ik}\right].$$

(3.26)

From (2.20) and (3.26), we have

$$\beta_m \overline{W}_{ijk}^h = \lambda_m \overline{W}_{ijk}^h+\mu_m\left(\delta_k^h g_{ij}-\delta_j^h g_{ik}\right)+\frac{1}{4}\left[R_k^h g_{ij}-R_j^h g_{ik}\right].$$

(3.27)

So, the proof of theorem is completed, we can say

Theorem3.2.7. The expansion derivative for the Berwald of projective curvature tensor $\overline{W}_{ijk}^h$ (2.20) satisfies the equation (3.27).

4. Conclusion

In this paper, we have studied new expansion identities for some curvature tensors in Finsler geometry. We have successfully proved several new identities that relate different curvature tensors and generalize some well-known results in Riemannian geometry.These results make a significant contribution to a deeper understanding of Finsler geometry, as they provide new tools for analyzing the structure of these spaces. Moreover, these identities can be useful in studying physical phenomena described by models based on Finsler geometry, such as generalized general relativity. Despite the progress made, there are still many open questions. For example, one can study similar identities for other curvature tensors, or in Finsler spaces with additional structures. Additionally, potential applications of these identities can be explored in fields such as string theory and brane theory.We hope that this research has made a valuable contribution to the field of Finsler geometry and will encourage other researchers to continue studying these rich and fascinating spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conficts of interest.

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