Abhath Journal of Basic and Applied Sciences., 1(2) (2022) 11-17

Abhath Journal of Basic and Applied Sciences

Website:https://ojs.abhath-ye.com/index.php/OJSRJBAS/about

doi:

Research paper, Short communication, Review, Technical paper

Existence and stability analysis to nonlocal implicit problems with ψ-piecewise fractional operators

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Received: 07 November 2022, Accepted: 15 December 2022, Published: 30 December 2022

Abstract

In this work, we investigate the existence, uniqueness, and Ulam-Hyers stability results for classes of nonlocal implicit problems involving ψ-piecewise Caputo fractional operators. Our approach is depends on fixed point theorems of Banach and Schaefer. Furthermore, the ensuring of the existence of solutions is shown by Ulam-Hyer's stability. At last, an example is given to show and approve our results.

***Keywords**: ψ-piecewise Caputo fractional problem; nonlocal condition; fixed point theorem.*

1. Introduction

It is noteworthy that scientists and researchers have given fractional calculus (FC) a lot of consideration. It is a result of its extensive range of applications across numerous fields and disciplines. FC's most important ideas and definitions have been discussed in [1] and [2]. The authors of [3] and [4], provided a brief history of fractional calculus as well as examples of its use in engineering and several scientific disciplines. Over the past few decades, numerous classes of fractional differential equations (FDEs) have undergone extensive research. For instance, ideas regarding the existence of unique solutions have been confirmed [5-7]. To solve these equations, numerical and analytical techniques have been developed [8-10]. These equations have demonstrated remarkable success in representing a variety of real-world problems. A key component of the theory of FDEs is the qualitative characteristics of solutions. For classical differential equations, the previously mentioned region has received extensive research. However, there are several FDE-related factors that need more research and analysis. By using Riemann-Liouville (R-L), Caputo, Hilfer, and other FDs, the emphasis on the existence and uniqueness has been further sharpened; for more information, see [11-17]. In this context, Agarwal et al. [18] established the existence results of the following Caputo-type FDE

$$\begin{cases} {}^{C}\mathbb{D}^{\mu}_{0^+}x(\zeta) = f(\zeta, x(\zeta)), \ \zeta \in [0,T], \ 0<\mu<1, \\ x(0)+g(x) = x_0. \end{cases} \tag{1}$$

By Kucche et al. [19], the fundamental theory of implicit FDEs with Caputo FD has been examined. In (2017) Almedia [20] suggested ψ -Caputo FD and gave good characteristics about this operator. The following nonlocal implicit FDE with ψ -Caputo FD was considered by Abdo et al. [11]

$$\begin{cases} \mathbb{D}^{\mu;\psi}_{a^+}x(\zeta) = f(\zeta, x(\zeta), \mathbb{D}^{\mu;\psi}_{a^+}x(\zeta)), \ \zeta \in [a,T], \ 0<\mu<1, \\ x(a)+g(x) = x_a. \end{cases} \tag{2}$$

$$\begin{cases} {}^{PC}\mathbb{D}^{\mu;\psi}_{0^+}x(\zeta) = \Theta(\zeta, x(\zeta), {}^{PC}\mathbb{D}^{\mu;\psi}_{0^+}x(\zeta)), \\ x(0) = x_0, \end{cases} \tag{3}$$

and

$$\begin{cases} {}^{PC}\mathbb{D}^{\mu;\psi}_{0^+}x(\zeta) = \Theta(\zeta, x(\zeta), {}^{PC}\mathbb{D}^{\mu;\psi}_{0^+}x(\zeta)), \\ x(0)+g(x) = x_0, \end{cases} \tag{4}$$

Where $0<\mu\leq 1$, $\zeta \in \mathbb{J} := [0,\tau]$, $x_0 \in \mathbb{R}$, $\Theta \in \mathcal{C}(\mathbb{J}\times\mathbb{R},\mathbb{R})$ $g \in \mathcal{C}(\mathbb{J},\mathbb{R})$ and ${}^{PC}\mathbb{D}^{\mu;\psi}_{0^+}$ is the ψ -piecewise Caputo FD given by

$${}^{PC}\mathbb{D}^{\mu;\psi}_{0^+}x(\zeta) = \begin{cases} \mathbb{D}^{1;\psi}x(\zeta) : \textit{if } \zeta \in [0,\zeta_1], \\ {}^{C}\mathbb{D}^{\mu;\psi}_{\zeta_1}x(\zeta) : \textit{if } \zeta \in [\zeta_1,\tau], \end{cases}$$

where $\mathbb{D}^{1;\psi}x(\zeta) := \frac{x'(\zeta)}{\psi'(\zeta)}$ is ψ -derivative on $0 \leq \zeta \leq \zeta_1$, and

$$ {}^{C}\mathbb{D}_{\zeta_1}^{\mu;\psi}\mathrm{x}(\zeta) = \frac{1}{\Gamma(1-\mu)}\int_{\zeta_1}^{\zeta}\psi'(t)(\psi(\zeta)-\psi(t))^{-\mu}\,\mathbb{D}^{1;\psi}\mathrm{x}(t)dt $$

is ψ -Caputo FD on $\zeta_1 \leq \zeta \leq \tau$.

It is crucial to keep in mind that using the nonlocal condition $\mathrm{x}(0) + g(\mathrm{x}) = \mathrm{x}_0$ instead of the initial condition $\mathrm{x}(0) = \mathrm{x}_0$ has a better influence on physical concerns (see [24]). We focus on the subject of innovative piecewise operators. As far as we are aware, not a lot of results using the piecewise FC have addressed the qualitative aspects of the aforementioned problems. The existence, uniqueness, and Ulam-Hyers stability results of the problems (3)-(4) based on common fixed point theorems related to Banach-type and Schauder-type will therefore be examined in order to close this gap.

Remark

- *If we set* $g(\mathrm{x}) \equiv 0,$ *then problem (4) reduces to (3).*
- *In (4), if we replace* ${}^{C}\mathbb{D}_{0^+}^{\mu;\psi}$ *instead of* ${}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}$, *then (4) reduces to (2) which was studied by Abdo et. [11], and by Benchohra and Bouriah [21] with* $\psi(\zeta) = \zeta$.

Our most recent outcomes for problem (4) are still accessible on (3).

The following is how this paper's main body is organized: A few necessary results and the basics of piecewise FC are presented in Section 2. In Section 3, we demonstrate our main findings regarding the issue (eq1). We summarize our research in the final part toward the finish.

2. Preliminary Results

We discuss some piecewise FC notions in this section. Let

$$ \mathcal{C} \coloneqq \mathcal{C}(\mathbb{J},\mathbb{R}) = \left\{\varpi : \mathbb{J} \to \mathbb{R};\ \|\varpi\| = \max_{\zeta\in\mathbb{J}}|\varpi(\zeta)|\right\} $$

Obviously $\mathcal{C}$ is a Banach space under $\|\varpi\|$.

Definition 2.1 [22] Let $\varpi : \mathbb{J} \to \mathbb{R}$ be a continuous and $\mu > 0$ be a number. The ψ -RL integral's piecewise representation is therefore provided by

$$ {}^{PRL}\mathbb{I}_{0^+}^{\mu;\psi}\varpi(\zeta) = \begin{cases} \mathbb{I}^{1;\psi}\ \varpi(\zeta), & \text{if } \zeta \in [0,\zeta_1], \\ {}^{RL}\mathbb{I}_{\zeta_1}^{\mu;\psi}\varpi(\zeta) & \text{if } \zeta \in [\zeta_1,\tau]. \end{cases} $$

where $\mathbb{I}^{1;\psi}\ \varpi(\zeta) = \int_0^{\zeta_1}\psi'(\zeta)\varpi(\zeta)d\zeta$ and ${}^{RL}\mathbb{I}_{\zeta_1}^{\mu;\psi}\varpi(\zeta) = \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta}\psi'(t)(\psi(\zeta)-\psi(t))^{\mu-1}\varpi(t)dt.$

Definition 2.2 [22] For $0 < \mu \leq 1,$ and $\varpi : \mathbb{J} \to \mathbb{R}$ be a continuous. The piecewise ψ -Caputo derivative is thus given by

$$ {}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}\varpi(\zeta) = \begin{cases} \mathbb{D}^{1;\psi}\varpi(\zeta), & \text{if } \zeta \in [0,\zeta_1], \\ {}^{C}\mathbb{D}_{\zeta_1}^{\mu;\psi}\varpi(\zeta) & \text{if } \zeta \in [\zeta_1,\tau], \end{cases} $$

where $\mathbb{D}^{1;\psi} = \frac{\varpi'}{\psi'}$ *and* ${}^{C}\mathbb{D}_{\zeta_1}^{\mu;\psi}$ *is defined by (z1).*

Lemma 2.3 *[22] Let* $0 < \mu \leq 1,$ *and* $\phi(0) = 0.$ *Then the following problem*

$$ {}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}\varpi(\zeta) = \phi(\zeta) $$
$$ \varpi(0) = \zeta_0 $$

has the following solution

$$ \varpi(\zeta) = \begin{cases} \varpi(0) + \int_0^{\zeta_1}\psi'(t)\phi(t)dt, & \text{if } \zeta \in [0,\zeta_1], \\ \varpi(\zeta_1) + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta}\psi'(t)(\psi(\zeta)-\psi(t))^{\mu-1}\phi(t)dt & \text{if } \zeta \in [\zeta_1,\tau]. \end{cases} $$

Lemma 2.4 *[22] Let* $0 < \mu \leq 1,$ *and* $\varpi \in \mathcal{C}$ *. Then*

$$ {}^{PRL}\mathbb{I}_{0^+}^{\mu;\psi}\,{}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}\varpi(\zeta) = \begin{cases} \mathbb{I}^{1;\psi}\,\mathbb{D}^{1;\psi}\ \varpi(\zeta) = \varpi(\zeta) - \varpi(0), \text{if } \zeta \in [0,\zeta_1], \\ {}^{RL}\mathbb{I}_{\zeta_1}^{\mu;\psi}\,{}^{C}\mathbb{D}_{\zeta_1}^{\mu;\psi}\varpi(\zeta) = \varpi(\zeta) - \varpi(\zeta_1), \text{if } \zeta \in [\zeta_1,\tau]. \end{cases} $$

We require both Schauder's [25] and Banach's [26] fixed point theories and generalized Gronwall's lemma [27] in order to complete our task.

3. Main Results

Here, we provide some qualitative analyses of the problem (4).

Lemma 3.1 Let $\Theta(\zeta,\mathrm{x},\omega) : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the problem (4) is corresponds to

$$ \mathrm{x}(\zeta) = \begin{cases} \mathrm{x}_0 - g(\mathrm{x}) + \int_0^{\zeta_1}\psi'(t)\Theta_{\mathrm{x}}(t)dt & \text{if } \zeta \in [0,\zeta_1], \\ \mathrm{x}(\zeta_1) - g(\mathrm{x}) + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta}\mathcal{K}_{\psi}^{\mu-1}(\zeta,t)\Theta_{\mathrm{x}}(t)dt, & \text{if } \zeta \in [\zeta_1,\tau], \end{cases} \quad \textbf{(5)} $$

where $\mathcal{K}_{\psi}^{\mu-1}(\zeta,t) \coloneqq \psi'(t)(\psi(\zeta)-\psi(t))^{\mu-1},$ and $\Theta_{\mathrm{x}} \in \mathcal{C}$ satisfies

$$\Theta_x(\zeta) = \begin{cases} \Theta\Big(\zeta, x_0 - g(x) + \int_0^{\zeta_1} \psi'(t)\Theta_x(t)dt, \Theta_x(\zeta)\Big) & if\ \zeta \in [0,\zeta_1], \\ \Theta\Big(\zeta, x(\zeta_1) - g(x) + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_x(t)dt, \Theta_x(\zeta)\Big), & if\ \zeta \in [\zeta_1,\tau]. \end{cases} \tag{6}$$

Proof: Let ${}^{PC}\mathbb{D}_{0^+}^{\mu;\psi} x(\zeta) = \Theta_x(\zeta)$. Applying ${}^{PRL}\mathbb{I}_{0^+}^{\mu;\psi}$, we obtain

$${}^{PRL}\mathbb{I}_{0^+}^{\mu;\psi}\, {}^{PC}\mathbb{D}_{0^+}^{\mu;\psi} x(\zeta) = {}^{PRL}\mathbb{I}_{0^+}^{\mu;\psi} \Theta_x(\zeta).$$

From Lemma 2.4, we obtain

$$\begin{cases} x(\zeta) = x(0) + \int_0^{\zeta_1} \psi'(t)\Theta_x(t)dt, \text{ for } \zeta \in [0,\zeta_1], \\ x(\zeta) = x(\zeta_1) + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_x(t)dt, \text{ for } \zeta \in [\zeta_1,\tau]. \end{cases}$$

In both instances, by applying the nonlocal condition, we obtain

$$x(\zeta) = \begin{cases} x_0 - g(x) + \int_0^{\zeta_1} \psi'(t)\Theta_x(t)dt, & if\ \zeta \in [0,\zeta_1], \\ x(\zeta_1) - g(x) + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_x(t)dt, & if\ \zeta \in [\zeta_1,\tau], \end{cases}$$

which lead to (5). Let (6), on the other hand, be satisfied. Set

$$x(\zeta) = \begin{cases} x_0 - g(x) + \int_0^{\zeta_1} \psi'(t)\Theta_x(t)dt & if\ \zeta \in [0,\zeta_1], \\ x(\zeta_1) - g(x) + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_x(t)dt, & if \zeta \in [\zeta_1,\tau]. \end{cases} \tag{7}$$

Applying ${}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}$ on (7), we have

$${}^{PC}\mathbb{D}_{0^+}^{\mu;\psi} x(\zeta) = \begin{cases} \mathbb{D}^{1;\psi}\Big(x_0 - g(x) + \int_0^{\zeta_1} \psi'(t)\Theta_x(t)dt\Big) & if\ \zeta \in [0,\zeta_1], \\ {}^{C}\mathbb{D}_{\zeta_1}^{\mu;\psi}\Big(x(\zeta_1) - g(x) + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_x(t)dt\Big), & if\ \zeta \in [\zeta_1,\tau], \end{cases}$$

Since

$$\mathbb{D}^{1;\psi}\, \mathbb{I}^{1;\psi}\Theta_x(\zeta) = \frac{1}{\psi'(\zeta)}\frac{d}{d\zeta}\int_0^{\zeta_1} \psi'(t)\Theta_x(t)dt = \Theta_x(\zeta)$$

on $0 \le \zeta \le \zeta_1$, and ${}^{C}\mathbb{D}_{\zeta_1}^{\mu;\psi}\mathbb{I}_{\zeta_1}^{\mu;\psi}\Theta_x(\zeta) = \Theta_x(\zeta)$ on $\zeta_1 \le \zeta \le \tau$, we obtain ${}^{PC}\mathbb{D}_{0^+}^{\mu;\psi} x(\zeta) = \Theta_x(\zeta)$, and hence

$${}^{PC}\mathbb{D}_{0^+}^{\mu;\psi} x(\zeta) = \Theta(\zeta, x(\zeta), {}^{PC}\mathbb{D}_{0^+}^{\mu;\psi} x(\zeta)), \text{ for all } \zeta \in \mathbb{J}.$$

The following presumptions will be used in the follow-up:

(A1) $\Theta : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Omega : \mathbb{R}^+ \to (0,\infty)$, and $\varphi_1, \varphi_2 : \mathbb{J} \to \mathbb{R}$ are continuous with Ω is an nondecreasing such that

$$|\Theta(\zeta,x,\omega)| \le \varphi_1(\zeta)\Omega(|x|) + \varphi_2(\zeta)|\omega|, \text{ for each } (\zeta,x,\omega) \in \mathbb{J}\times\mathbb{R}\times\mathbb{R}.$$

(A2) $g : \mathcal{C} \to \mathbb{R}$ is continuous and compact with $|g(x)| \le a|x| + b$, for $x \in \mathcal{C}$, $a,b > 0$.

(A3) There exist $\kappa_1, \kappa_2 > 0$ such that $0 < \kappa_1, \kappa_2 < 1$, and

$$|\Theta(\zeta,x,\omega) - \Theta(\zeta,\bar{x},\bar{\omega})| \le \kappa_1|x-\bar{x}| + \kappa_2|\omega-\bar{\omega}|, \text{ for each } \zeta \in \mathbb{J},\ x,\omega,\bar{x},\bar{\omega} \in \mathbb{R}.$$

(A4) There exists $\kappa_3 > 0$ such that $0 < \kappa_3 < 1$ and $|g(x) - g(\omega)| \le \kappa_3|x-\omega|$, for $x, \omega \in \mathcal{C}$.
We will now use Schauder's theorem to demonstrate the existence theorem for (4).

Theorem 3.2 Let (A1) and (A2) hold. Then the piecewise problem (4) has a least one solution on $\mathbb{J}$.

Proof. Consider the operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ such that $(\mathcal{Q}x)(\zeta) = x(\zeta)$, *i.e.*,

$$(\mathcal{Q}x)(\zeta) = \begin{cases} x_0 - g(x) + \int_0^{\zeta_1} \psi'(t)\Theta_x(t)dt & if\ \zeta \in [0,\zeta_1], \\ x(\zeta_1) - g(x) + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_x(t)dt, & if \zeta \in [\zeta_1,\tau]. \end{cases}$$

where $\Theta_x \in \mathcal{C}$ with $\Theta_x(\zeta) := \Theta(\zeta, x(\zeta), \Theta_x(\zeta))$.
Set $\mathcal{S}_\beta = \{x \in \mathcal{C} : \|x\|_{\mathcal{C}} \le \beta\}$ with $\beta \ge \max\{\beta_1,\beta_2\}$, where

$$\beta_1 := |x_0| + a\beta + b + \frac{\varphi^\star\Omega(\beta)}{1-\psi^\star}[\psi(\zeta_1) - \psi(0)],$$

$$\beta_2 := |x(\zeta_1)| + a\beta + b + \frac{\varphi^\star\Omega(\beta)}{1-\psi^\star}\frac{([\psi(b) - \psi(\zeta_1)])^\mu}{\Gamma(\mu+1)},$$

$\varphi^\star = \sup|\varphi_1(\zeta)|$, and $\psi^\star = \sup|\varphi_2(\zeta)|$, with $0 < \psi^\star < 1$.

For any $x \in \mathcal{S}_\beta$, and by (A1), we have

$$\begin{aligned} |\Theta_x(\zeta)| &= |\Theta(\zeta, x(\zeta), \Theta_x(\zeta))| \\ &\le \varphi_1(\zeta)\Omega(\|x\|_{\mathcal{C}}) + \varphi_2(\zeta)|\Theta_x(\zeta)| \\ &\le \varphi^\star\Omega(\beta) + \psi^\star\|\Theta_x\|_{\mathcal{C}}. \end{aligned}$$

As $\psi^\star < 1$, we get

$$\|\Theta_x\|_{\mathcal{C}} \le \frac{\varphi^\star\Omega(\beta)}{1-\psi^\star}. \tag{8}$$

Consequently, we carry out the below steps:

Step 1: $\mathcal{Q}(\mathcal{S}_\beta)$ is bounded.

Case 1: For $\zeta \in [0,\zeta_1]$, we have

$$
\begin{aligned}
|(Q\varkappa)(\zeta)| &\le |\varkappa_0| + \sup_{\varkappa\in\mathcal{S}_\beta}|g(\varkappa)| + \sup_{\zeta\in[0,\zeta_1]}\int_0^{\zeta_1}\psi'(t)|\Theta_\varkappa(t)|dt \\
&\le |\varkappa_0| + a\|\varkappa\|_{\mathcal{C}} + b + \frac{\varphi^\star\Omega(\beta)}{1-\psi^\star}[\psi(\zeta_1)-\psi(0)] \\
&\le \beta_1 \le \beta.
\end{aligned} \tag{9}
$$

Case 2: For $\zeta \in [\zeta_1, \tau]$, we have

$$
\begin{aligned}
|(Q\varkappa)(\zeta)| &\le \sup_{\zeta\in[\zeta_1,\tau]}|\varkappa(\zeta_1)| + \sup_{\varkappa\in\mathcal{S}_\beta}|g(\varkappa)| + \frac{1}{\Gamma(\mu)}\sup_{\zeta\in[\zeta_1,\tau]}\int_{\zeta_1}^{\zeta}\mathcal{K}_\psi^{\mu-1}(\zeta,t)|\Theta_\varkappa(t)|dt \\
&\le |\varkappa(\zeta_1)| + a\|\varkappa\|_{\mathcal{C}} + b + \frac{\varphi^\star\Omega(\beta)}{1-\psi^\star}\frac{(\psi(b)-\psi(\zeta_1))^\mu}{\Gamma(\mu+1)} \\
&\le \beta_2 \le \beta.
\end{aligned} \tag{10}
$$

(9) and (10) give $\|Q\varkappa\|_{\mathcal{C}} \le \beta.$ Thus $Q(\mathcal{S}_\beta) \subset \mathcal{S}_\beta$. Since $\mathcal{S}_\beta$ is bounded, $Q(\mathcal{S}_\beta)$ is bounded.

Step 2: $Q : \mathcal{S}_\beta \to \mathcal{S}_\beta$ is continuous. Let a sequence $(\varkappa_n)$ such that $\varkappa_n \to \varkappa$ in $\mathcal{S}_\beta$ as $n \to \infty$. Then for $\zeta \in [0, \zeta_1]$, we have

$$|(Q\varkappa_n)(\zeta) - (Q\varkappa)(\zeta)| \le |g(\varkappa_n) - g(\varkappa)| + \int_0^{\zeta_1}\psi'(t)|\Theta_{\varkappa_n}(t) - \Theta_\varkappa(t)|dt,$$

and for $\zeta \in [\zeta_1, \tau]$, we have

$$
\begin{aligned}
|(Q\varkappa_n)(\zeta) - (Q\varkappa)(\zeta)| &\le |\varkappa_n(\zeta_1) - \varkappa(\zeta_1)| + |g(\varkappa_n) - g(\varkappa)| \\
&\quad + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta}\mathcal{K}_\psi^{\mu-1}(\zeta,t)|\Theta_{\varkappa_n}(t) - \Theta_\varkappa(t)|dt,
\end{aligned}
$$

where $\Theta_\varkappa, \Theta_{\varkappa_n} \in \mathcal{C}$, with $\Theta_{\varkappa_n}(\zeta) := \Theta(\zeta, \varkappa_n(\zeta), \Theta_{\varkappa_n}(\zeta))$ and $\Theta_\varkappa(\zeta) := \Theta(\zeta, \varkappa(\zeta), \Theta_\varkappa(\zeta))$. Since $\varkappa_n \to \varkappa$ as $n \to \infty$ and $\Theta_\varkappa, \Theta_{\varkappa_n}$, Θ and g are continuous, the Lebesgue dominated convergence theorem gives that

$$\|Q\varkappa_n - Q\varkappa\|_{\mathcal{C}} \to 0, \; as \;\; n \to \infty.$$

Step 3: $Q(\mathcal{S}_\beta)$ is equicontinuous. Let $\zeta \in [0, \zeta_1]$ with $\zeta_m < \zeta_n \in [0, \zeta_1]$, we have

$$|(Q\varkappa)(\zeta_n) - (Q\varkappa)(\zeta_m)| \le |g(\varkappa(\zeta_n)) - g(\varkappa(\zeta_m))|.$$

Let $\zeta \in [\zeta_1, \tau]$ with $\zeta_m < \zeta_n \in [\zeta_1, \tau]$. Then

$$
\begin{aligned}
&|(Q\varkappa)(\zeta_n) - (Q\varkappa)(\zeta_m)| \\
&\le |g(\varkappa(\zeta_n)) - g(\varkappa(\zeta_m))| \\
&+ \left|\frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta_n}\mathcal{K}_\psi^{\mu-1}(\zeta_n,t)\Theta_\varkappa(t)dt - \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta_m}\mathcal{K}_\psi^{\mu-1}(\zeta_m,t)\Theta_\varkappa(t)dt\right| \\
&\le |g(\varkappa(\zeta_n)) - g(\varkappa(\zeta_m))| + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta_m}\left|\mathcal{K}_\psi^{\mu-1}(\zeta_m,t) - \mathcal{K}_\psi^{\mu-1}(\zeta_n,t)\right||\Theta_\varkappa(t)|dt \\
&+ \frac{1}{\Gamma(\mu)}\int_{\zeta_m}^{\zeta_n}\mathcal{K}_\psi^{\mu-1}(\zeta_n,t)|\Theta_\varkappa(t)|dt \\
&\le |g(\varkappa(\zeta_n)) - g(\varkappa(\zeta_m))| - \frac{(\psi(\zeta_m)-\psi(\zeta_1))^\mu + (\psi(\zeta_n)-\psi(\zeta_m))^\mu + (\psi(\zeta_n)-\psi(\zeta_1))^\mu}{\Gamma(\mu+1)}\frac{\varphi^\star\Omega(\beta)}{1-\psi^\star} \\
&+ \left(\frac{(\psi(\zeta_n)-\psi(\zeta_m))^\mu}{\Gamma(\mu+1)}\right)\frac{\varphi^\star\Omega(\beta)}{1-\psi^\star} \\
&\le |g(\varkappa(\zeta_n)) - g(\varkappa(\zeta_m))| + \frac{2(\psi(\zeta_n)-\psi(\zeta_m))^\mu}{\Gamma(\mu+1)}\frac{\varphi^\star\Omega(\beta)}{1-\psi^\star}.
\end{aligned}
$$

Since g is continuous and compact,

$|g(\varkappa(\zeta_n)) - g(\varkappa(\zeta_m))| \to 0$, as $\zeta_m \to \zeta_n$. It follows from (q3) and (eq6) that

$$|(Q\varkappa)(\zeta_n) - (Q\varkappa)(\zeta_m)| \to 0, \text{as } \zeta_m \to \zeta_n.$$

As a result, Q is relatively compact on $\mathcal{S}_\beta$. Q is therefore completely continuous according to the Arzela-Ascolli theorem. Thus, Schauder's theorem demonstrates that problem (4) has at least one solution. Next, using Banach's theorem, we demonstrate the uniqueness theorem for (4).

Theorem 3.3 Let (A3)-(A4) hold. If $\max_{\zeta\in\mathbb{J}}\{\zeta_1, \zeta_2\} = \zeta < 1$, then the problem (4) has a unique solution on $\mathbb{J}$, where

$$\zeta_1 := \kappa_3 + \frac{\kappa_1}{1-\kappa_2}\psi(\zeta_1) - \psi(0), \text{ and}$$

$$\zeta_2 := \kappa_3 + \frac{\kappa_1}{1-\kappa_2}\frac{(\psi(\tau)-\psi(\zeta_1))^\mu}{\Gamma(\mu+1)}.$$

Proof: Consider $\varkappa$ and $\bar{\varkappa}$ in $\mathcal{C}$, then

$$
\begin{aligned}
|\Theta_\varkappa(\zeta) - \Theta_{\bar{\varkappa}}(\zeta)| &= |\Theta(\zeta, \varkappa(\zeta), \Theta_\varkappa(\zeta)) - \Theta(\zeta, \bar{\varkappa}(\zeta), \Theta_{\bar{\varkappa}}(\zeta))| \\
&\le \kappa_1|\varkappa(\zeta) - \bar{\varkappa}(\zeta)| + \kappa_2|\Theta_\varkappa(\zeta) - \Theta_{\bar{\varkappa}}(\zeta)|,
\end{aligned}
$$

which implies that

$$|\Theta_\varkappa(\zeta) - \Theta_{\bar{\varkappa}}(\zeta)| \le \frac{\kappa_1}{1-\kappa_2}|\varkappa(\zeta) - \bar{\varkappa}(\zeta)|.$$

Hence, we have two cases:

Case 1: For $\zeta \in [0, \zeta_1]$,

$$
\begin{aligned}
|(Q\varkappa)(\zeta) - (Q\bar{\varkappa})(\zeta)| &\le |g(\varkappa) - g(\bar{\varkappa})| + \int_0^{\zeta_1}\psi'(t)|\Theta_\varkappa(t) - \Theta_{\bar{\varkappa}}(t)|dt \\
&\le \left(\kappa_3 + \frac{\kappa_1(\psi(\zeta_1)-\psi(0))}{1-\kappa_2}\right)\|\varkappa - \bar{\varkappa}\|_{\mathcal{C}}.
\end{aligned}
$$

Case 2: For $\zeta \in [\zeta_1, \tau]$,

$$
\begin{aligned}
|(Q\varkappa)(\zeta) - (Q\bar{\varkappa})(\zeta)| &\le |g(\varkappa) - g(\bar{\varkappa})| + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta}\mathcal{K}_\psi^{\mu-1}(\zeta,t)|\Theta_\varkappa(t) - \Theta_{\bar{\varkappa}}(t)|dt \\
&\le \left(\kappa_3 + \frac{(\psi(\tau)-\psi(\zeta_1))^\mu}{\Gamma(\mu+1)}\frac{\kappa_1}{1-\kappa_2}\right)\|\varkappa - \bar{\varkappa}\|_{\mathcal{C}}.
\end{aligned}
$$

Consequently,

$$\|Q\varkappa - Q\bar{\varkappa}\|_{\mathcal{C}} \le \zeta\|\varkappa - \bar{\varkappa}\|_{\mathcal{C}}.$$

As $\zeta < 1$, $\mathcal{Q}$ is a contraction. Thus, Banach's theorem demonstrates that (4) has a unique solution exists on $\mathbb{J}$.

4. UH Stability Analysis

In this section, we provide the problem's UH Stability (4).

Definition 4.1 (4) is UH stable if there exists a $K_f > 0$ such that $\forall \quad \epsilon > 0$ and each solution $\omega \in \mathcal{C}$ of the inequality

$$\left| {}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}\omega(\zeta) - \Theta_\omega(\zeta) \right| \le \epsilon, \quad \zeta \in \mathbb{J}, \tag{11}$$

there exists a solution $\chi \in \mathcal{C}$ of (4) satisfies

$$|\omega(\zeta) - \chi(\zeta)| \le K_f \epsilon,$$

where $\Theta_\omega(\zeta) := {}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}\omega(\zeta)$ and

$$\Theta_\omega(\zeta) = \Theta(\zeta, \omega(\zeta), \Theta_\omega(\zeta)).$$

Remark 4.2 $\omega \in \mathcal{C}$ satisfies (11) iff

(i) there exist function $\sigma \in \mathcal{C}$ with $|\sigma(\zeta)| \le \epsilon$, $\zeta \in \mathbb{J}$;

(ii) for all $\zeta \in \mathbb{J}$,

$${}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}\omega(\zeta) = \Theta_\omega(\zeta) + \sigma(\zeta),$$

Lemma 4.3 *Let* $0 < \mu \le 1$, *and* $\omega \in \mathcal{C}$ *is a solution of (11). Then* ω *satisfies*

$$\begin{cases} \left| \omega(\zeta) - \mathcal{Z}_0 - \int_0^{\zeta_1} \psi'(t)\Theta_\omega(t)dt \right| \le [\psi(\zeta_1) - \psi(0)]\epsilon, & \text{if } \zeta \in [0, \zeta_1], \\ \left| \omega(\zeta) - \mathcal{Z}_1 - \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_\omega(t)dt \right| \le \frac{[\psi(\tau)-\psi(\zeta_1)]^\mu}{\Gamma(\mu+1)}\epsilon, & \text{if } \zeta \in [\zeta_1, \tau], \end{cases}$$

where $\mathcal{Z}_1 = \omega(\zeta_1) - g(\omega)$. $\mathcal{Z}_0 = \omega_0 - g(\omega)$ *and*

Proof: Let ω is a solution of (11). By part (ii) of Remark 4.2, we have

$$\begin{cases} {}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}\omega(\zeta) = \Theta_\omega(\zeta) + \sigma(\zeta), \\ \omega(0) + g(\omega) = \omega_0. \end{cases} \tag{12}$$

The solution of (12) is

$$\omega(\zeta) = \begin{cases} \mathcal{Z}_0 + \int_0^{\zeta_1} \psi'(t)[\Theta_\omega(t) + \sigma(t)]dt, & \text{if } \zeta \in [0, \zeta_1], \\ \mathcal{Z}_1 + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)[\Theta_\omega(t) + \sigma(t)]dt, & \text{if } \zeta \in [\zeta_1, \tau]. \end{cases}$$

By (i) of Remark 4.2, we get

$$\left| \omega(\zeta) - \mathcal{Z}_0 - \int_0^{\zeta_1} \psi'(t)\Theta_\omega(t)dt \right| \le \int_0^{\zeta_1} \psi'(t)|\sigma(t)|dt \le [\psi(\zeta_1) - \psi(0)]\epsilon, \text{ for } \zeta \in [0, \zeta_1],$$

and

$$\left| \omega(\zeta) - \mathcal{Z}_1 - \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_\omega(t)dt \right| \le \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)|\sigma(t)|dt \le \frac{[\psi(\tau) - \psi(\zeta_1)]^\mu}{\Gamma(\mu+1)}\epsilon, \text{ for } \zeta \in [\zeta_1, \tau].$$

Theorem 4.4 Under the hypotheses of Theorem 3.2. The solution of the problem (4) is HU and GHU stable.

Proof: Let $\omega \in \mathcal{C}$ is a solution of (11), and $\chi \in \mathcal{C}$ is a unique solution of

$$\begin{cases} {}^{PC}\mathbb{D}_{0^+}^{\mu;\psi}\chi(\zeta) = \Theta_\chi(\zeta), \\ \chi(0) + g(\chi) = \omega(0) + g(\omega). \end{cases}$$

From Lemma 2.4, we get

$$\chi(\zeta) = \begin{cases} \mathcal{V}_0 + \int_0^{\zeta_1} \psi'(t)\Theta_\chi(t)dt, & \text{if } \zeta \in [0, \zeta_1], \\ \mathcal{V}_1 + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_\chi(t)dt, & \text{if } \zeta \in [\zeta_1, \tau], \end{cases} \tag{13}$$

where $\mathcal{V}_0 = \chi_0 - g(\chi)$ and $\mathcal{V}_1 = \chi(\zeta_1) - g(\chi)$.

Clearly, if $\chi(0) + g(\chi) = \omega(0) + g(\omega)$, then $\mathcal{V}_0 = \mathcal{Z}_0$, and $\mathcal{V}_1 = \mathcal{Z}_1$. Hence, (13) becomes

$$\chi(\zeta) = \begin{cases} \mathcal{Z}_0 + \int_0^{\zeta_1} \psi'(t)\Theta_\chi(t)dt, & \text{if } \zeta \in [0, \zeta_1], \\ \mathcal{Z}_1 + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_\chi(t)dt, & \text{if } \zeta \in [\zeta_1, \tau], \end{cases}$$

Using Lemma 4.3 and (A4), then for $\zeta \in [0, \zeta_1]$,

$$\begin{aligned} |\omega(\zeta) - \chi(\zeta)| &= \left| \omega(\zeta) - \mathcal{Z}_0 - \int_0^{\zeta_1} \psi'(t)\Theta_\chi(t)dt \right| \\ &\le \left| \omega(\zeta) - \mathcal{Z}_0 - \int_0^{\zeta_1} \psi'(t)\Theta_\omega(t)dt \right| + \int_0^{\zeta_1} \psi'(t)|\Theta_\omega(t) - \Theta_\chi(t)|dt \\ &\le [\psi(\zeta_1) - \psi(0)]\epsilon + \frac{\kappa_1}{1-\kappa_2}\int_0^{\zeta_1} \psi'(t)|\omega(t) - \chi(t)|dt. \end{aligned}$$

Using generalized Gronwall's Lemma [27], we obtain

$$\begin{aligned} |\omega(\zeta) - \chi(\zeta)| &\le \epsilon[\psi(\zeta_1) - \psi(0)]\exp\left(\int_0^{\zeta_1} \frac{\kappa_1}{1-\kappa_2}\psi'(t)\right)dt \\ &= \epsilon[\psi(\zeta_1) - \psi(0)]\exp\left(\frac{\kappa_1[\psi(\zeta_1) - \psi(0)]}{1-\kappa_2}\right) := \epsilon K_0. \end{aligned} \tag{14}$$

For $\zeta \in [\zeta_1, \tau]$, we have

$$\begin{aligned} |\omega(\zeta) - \chi(\zeta)| &= \left| \omega(\zeta) - \mathcal{W}_1 - \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_\chi(t)dt \right| \\ &\le \left| \omega(\zeta) - \mathcal{W}_1 - \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)\Theta_\omega(t)dt \right| \\ &\quad + \frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)|\Theta_\omega(t) - \Theta_\chi(t)|dt \\ &\le \frac{[\psi(b) - \psi(\zeta_1)]^\mu}{\Gamma(\mu+1)}\epsilon + \frac{\kappa_1}{1-\kappa_2}\frac{1}{\Gamma(\mu)}\int_{\zeta_1}^{\zeta} \mathcal{K}_\psi^{\mu-1}(\zeta,t)|\omega(t) - \chi(t)|dt. \end{aligned}$$

Using generalized fractional Gronwall's Lemma [27], we obtain

$$\begin{aligned} |\omega(\zeta) - \chi(\zeta)| &\le \frac{[\psi(b) - \psi(\zeta_1)]^\mu}{\Gamma(\mu+1)}\epsilon E_\mu\left(\frac{\kappa_1\Gamma(\mu)}{1-\kappa_2}[\psi(\zeta) - \psi(0)]^\mu\right) \\ &\le \frac{[\psi(b) - \psi(\zeta_1)]^\mu}{\Gamma(\mu+1)}\epsilon E_\mu\left(\frac{\kappa_1\Gamma(\mu)}{1-\kappa_2}[\psi(\tau) - \psi(0)]^\mu\right) \\ &:= \epsilon K_1. \end{aligned} \tag{15}$$

It follows from (14) and (15) that

$$|\omega(\zeta)-\varkappa(\zeta)| \leq \begin{cases} K_0\epsilon, for\, \zeta \in [0,\zeta_1] \\ K_1\epsilon, for\, \zeta \in [\zeta_1,\tau] \end{cases}, \tag{16}$$

$$K_0 = [\psi(\zeta_1)-\psi(0)]\exp\left(\frac{\kappa_1[\psi(\zeta_1)-\psi(0)]}{1-\kappa_2}\right), \text{ and}$$

$$K_1 = \frac{[\psi(b)-\psi(\zeta_1)]^\mu}{\Gamma(\mu+1)}E_\mu\left(\frac{\kappa_1\Gamma(\mu)}{1-\kappa_2}[\psi(\tau)-\psi(0)]^\mu\right)$$

Hence, the problem (4) is UH stable in $\mathcal{C}$. Moreover, if there exists a nondecreasing function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ such that $\varphi(\epsilon)=\epsilon$. Then from (16), we have

$$|\omega(\zeta)-\varkappa(\zeta)| \leq \begin{cases} K_0\varphi(\epsilon), for\, \zeta \in [0,\zeta_1] \\ K_1\varphi(\epsilon), for\, \zeta \in [\zeta_1,\tau] \end{cases}$$

with $\varphi(0)=0$, which proves the problem (4) is GUH stable in $\mathcal{C}$.

5. An Example

In this section, we give an examples to illustrate the main results.

Consider the following piecewise problem

$$\begin{cases} {}^{PC}\mathbb{D}_{0^+}^{\frac{1}{3};\psi}\varkappa(\zeta) = \Theta\left(\zeta,\varkappa(\zeta),{}^{PC}\mathbb{D}_{0^+}^{\frac{1}{3};\psi}\varkappa(\zeta)\right),\ \zeta\in[0,1], \\ \varkappa(0)+\sum_{i=1}^n c_i\varkappa(\zeta_i) = \frac{1}{4}, \end{cases} \tag{17}$$

or

$$\begin{cases} \varkappa'(\zeta) = \psi'(\zeta)\Theta(\zeta,\varkappa(\zeta),\varkappa'(\zeta)), \quad \zeta\in[0,\frac{1}{2}], \\ {}^{C}\mathbb{D}_{\frac{1}{2}^+}^{\frac{1}{3},\psi}\varkappa(\zeta) = \Theta\left(\zeta,\varkappa(\zeta),{}^{C}\mathbb{D}_{\frac{1}{2}^+}^{\frac{1}{3}}\varkappa(\zeta)\right),\ if\ \zeta\in[\frac{1}{2},1], \\ \varkappa(0)+\sum_{i=1}^n c_i\varkappa(\zeta_i) = \frac{1}{4}, \end{cases}$$

where $\mu=\frac{1}{3}$, $\varkappa_0=\frac{1}{4}$, $0<\zeta_1=\frac{1}{2}<\ldots<\zeta_n<1=\tau$, and c_i are positive constants with $\sum_{i=1}^n c_i<\frac{1}{5}$. Set

$$\Theta(\zeta,\varkappa,\omega) = \frac{e^{-\zeta}}{(8+e^{\zeta})(2+|\varkappa|+|\omega|)},\ \zeta\in[0,1],\ \varkappa,\omega\in[0,\infty),$$

and

$$g(\varkappa) = \sum_{i=1}^n c_i\varkappa(\zeta_i),\ \varkappa\in[0,\infty),$$

Let $\varkappa,\omega,\bar{\varkappa},\bar{\omega}\in[0,\infty)$, $\zeta\in[0,1]$. Then

$$|f(\zeta,\varkappa,\omega)-f(\zeta,\bar{\varkappa},\bar{\omega})| \leq \frac{e^{-\zeta}}{(8+e^{\zeta})}\left|\frac{|\varkappa-\bar{\varkappa}|+|\omega-\bar{\omega}|}{(2+|\varkappa|+|\omega|)(2+|\bar{\varkappa}|+|\bar{\omega}|)}\right| \leq \frac{1}{9}|\varkappa-\bar{\varkappa}|+\frac{1}{9}|\omega-\bar{\omega}|.$$

Hence the condition (A3) holds with $\kappa_1=\kappa_2=\frac{1}{9}$. Also we have

$$|g(\varkappa)-g(\omega)| = \left|\sum_{i=1}^n c_i\varkappa(\zeta_i)-\sum_{i=1}^n c_i\omega(\zeta_i)\right| \leq \sum_{i=1}^n c_i|\varkappa-\omega| \leq \frac{1}{5}|\varkappa-\omega|$$

Hence the condition (A4) holds with $\kappa_3=\frac{1}{5}$. Moreover, the following condition

$$\max\{\zeta_1,\zeta_2\} = \max\left\{\kappa_3+\frac{\kappa_1}{1-\kappa_2}[\psi(\zeta_1)-\psi(0)],\kappa_3+\frac{\kappa_1}{1-\kappa_2}\frac{(\psi(\tau)-\psi(\zeta_1))^\mu}{\Gamma(\mu+1)}\right\} = \max\left\{\frac{1}{5}+\frac{\frac{1}{9}}{1-\frac{1}{9}}\frac{1}{6},\frac{1}{5}+\frac{1}{8\sqrt[3]{3}\,\Gamma(\frac{4}{3})}\right\} = \frac{1}{5}+\frac{1}{8\sqrt[3]{3}\,\Gamma(\frac{4}{3})}<1$$

is satisfied with $\psi'(\zeta)=\frac{\zeta}{3}$, $\zeta_1=\frac{1}{2}$, and $\tau=1$. Thus, with the assistance of Theorem 3.2, the problem (19) has a unique solution $[0,1]$.

Further, $1-\frac{\kappa_1(\psi(\zeta_1)-\psi(0))}{1-\kappa_2}=\frac{47}{48}<1$, and

$$1-\frac{\kappa_1}{1-\kappa_2}\frac{(\psi(\tau)-\psi(\zeta_1))^\mu}{\Gamma(\mu+1)} = 1-\frac{1}{192\sqrt[3]{3}\,\Gamma(\frac{4}{3})}<1.$$

It follows that

$$K_0 = \zeta_1\exp\left(\frac{\kappa_1(\psi(\zeta_1)-\psi(0))}{1-\kappa_2}\right) = \frac{1}{2}e^{\frac{1}{48}}>0$$

and

$$K_1 = \frac{(\psi(\tau)-\psi(\zeta_1))^\mu}{\Gamma(\mu+1)}\left(\frac{\kappa_1}{(1-\kappa_2)}+\frac{1}{\Gamma(\mu+1)}\right) = \frac{\frac{1}{8}+\frac{1}{\Gamma(\frac{4}{3})}}{\sqrt[3]{3}\,\Gamma(\frac{4}{3})}>0,$$

which implies that the problem (19) is HU stable.

6. Conclusions

Recently, a variety of approaches have been put out to depict the behaviors of some complex global challenges that are occurring in a variety of academic domains. One of these issues is the multi-step behavior that several of these issues exhibit. In this context, Atangana and Araz [19] proposed the idea of the piecewise derivative. For the implicit problems (3) and (4) with piecewise ψ-Caputo FD, the existence, uniqueness, and UH stability results have been obtained as an additional contribution to this topic. Our methodology for this work has been based on Gronwall's Lemma, the fixed point theorem of Banach and Schaefer, and these two theorems. Finally, we have produced an example illustration to verify the findings.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was conducted during our work at Hodeidah University.

How to Cite : Abdo M. S. (2022). Existence and stability analysis to nonlocal implicit problems with ψ-piecewise fractional operators, *Abhath Journal of Basic and Applied Sciences*, 1(2), 11-17.

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