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Contents of the Issue

Introduction of the Issue

We are pleased and delighted to present the researchers with this issue of the 'Abhath' Journal of Basic and Applied Sciences, which is the first issue of the first volume, the issuance of which emanates as an affirmation of moving forward towards issuing specialized quality journals.

The Faculty of Education at Hodeidah University aims, by issuing this journal, to publish specialized researches in basic and applied sciences, from inside and outside Yemen, in the English language.

On this occasion, the journal invites male and female researchers to submit their researches for publication in the next issues of the journal.

In conclusion, the editorial board of the journal extends its thanks and gratitude to Prof. Mohammed Al-Ahdal – Rector of the university – the general supervisor of the journal, for his support and encouragement for the establishment of this journal. Furthermore, thanks are extended to Prof. Mohammed Bulghaith – University Vice-Rector for Higher Studies and Scientific Research – vice-supervisor of the journal, for his cooperation in facilitating the procedures for the issuance of this issue. Nevertheless, thanks are for all researchers whose scientific articles were published in this issue, and for the editorial board of the journal, which worked tirelessly to produce this issue in this honorable way.

Journal Chief Editor

Prof. Yusuf Al-Ojaily

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Boundary value problem for fractional neutral differential equations with infinite delay

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Abstract

In this article, we develop and extend some qualitative analyses of a class of neutral functional differential equations involving a Caputo fractional derivative over the infinite delay period. The existence and uniqueness results are proved based on an equivalent fractional integral equation with the help of Banach contraction principle and Schauder's fixed point theorem.

Keywords: Caputo fractional derivative; Neutral functional differential equation; Existence of solution; Fixed point theorem.

Introduction

The existence of solutions to boundary value problems (BVPs) for fractional functional differential equations (FFDEs) with finite delay have been extensively studied, see [4-8,13] and references therein. However, research for the existence of solutions for the BVPs of FDEs with infinite delay proceeded very slowly. Recently, Yong et al., in [12] investigated the existence of positive solution for floquet BVP concerning FFDE with finite delay

$$^{c}\mathbf{D}_{0^{+}}^{\alpha}y(t)=f(t,y_{t}),\quad t\in[0,b],$$
$$Ay_{0}-y_{b}=\phi,$$

where $A>1$ and $b>0$ with $0<\alpha\leq 1$, $^{c}\mathbf{D}_{0^{+}}^{\alpha}$ is the Caputo's fractional derivative of order α, $f:[0,b]\times C[-r,0]\rightarrow \mathbb{R}$ is a given function satisfying some assumptions, and $\phi\in C[-r,0]$. Their results were obtained by using two fixed point theorems (FPTs) on appropriate cones.

Neutral FFDEs basically appeared as models of electrical networks arising in high-speed computer systems, these problems are used to interconnect switching circuits. For more details on such Neutral FFDEs, we advise looking at the following papers [8,9,11,14-19,21], and references therein.

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Motivated by the above works, and inspired by [12], this paper is concerned with the existence and uniqueness of the solution of the BVP for nonlinear neutral FFDEs with an infinite delay of the type

$$ {}^{c}\mathbf{D}_{0^+}^{\alpha}[y(t)-g(t,y_t)] = f(t,y_t), \quad t \in J_1 = [0,b], \qquad (1)$$

$$\beta y_0 - y_b = \phi \in \mathbf{B}, \qquad \text{on } J_2 = (-\infty,0], \qquad (2)$$

where $0<\alpha\le 1,\ \beta>1,\ b>0,\ {}^{c}\mathbf{D}_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha,\ f,g : J_1\times\mathbf{B}\to R$ are appropriate functions contains some hypotheses that will be specified later, $\phi : J_2 \to \mathrm{R}$ in the phase space $\mathbf{B}$, and $y_t(\zeta) = y(t+\zeta)$ for $\zeta \le 0$.

In this paper, our main objectives are highlighted as follows:

- Study the existence and uniqueness results of (1)-(2) with infinite delay.
- We deduce that the equivalent operator has a fixed point, it means that the problem (1)-(2) has one solution, which is also a unique solution.
- The used techniques to demonstrate the existence and uniqueness results are a variety of tools such as fractional calculus, Hólder inequality, Lebesgue dominated convergence theorem, Arzelá-Ascoli theorem, Banach's and Schauder's FPTs (fixed point theorems).

Preliminaries

This section is devoted to recall some notaion, basic definitions and preliminary facts from fractional calculus theory and nonlinear analysis which will be used throughout this paper.

Let $J_1 := [0,b],\ J_2 := (-\infty,0]$, and $J := (-\infty,b],\ (b>0).$ We denote by $C(J_1,\mathrm{R})$ the Banach space of all continuous real functions defined on J_1 endowed with the norm $\|y\|_C = \sup\{|y(t)| : t\in J_1\}$ for any $y\in C(J_1,\mathrm{R})$, and let $\mathbf{L}^p(J_1,\mathrm{R})\ (1\le p<\infty)$ the set of those Lebesgue measurable functions y on J_1 such that

$$\|y\|_{\mathbf{L}^p} = \left(\int_{J_1} |y(t)|^p\,dt\right)^{\frac{1}{p}} < \infty.$$

For any $t\in J_1$ and $y : J\to\mathrm{R}$ is continuous, we denote by $y_t : J_2 \to \mathrm{R}$ the element of phase space $\mathbf{B}$ defined as

$y_t(\zeta) = y(t+\zeta)$, for all $\zeta \in J_2$.

Also we consider the space

$\mathbf{G}_4 = \{y \ : \ J \to \mathrm{R};\ y\,|_{J_2} \in \mathbf{B},\ y\ \ |_{J_1} \in C(J_1, \mathrm{R})\}$,

where $y\,|_{J_1}$ is the restriction of y to J_1.

Definition2.1. *[2]. Let* $\alpha > 0$ *be a fixed number. The left sided Riemann-Liouville fractional integral of order* α *of a function* $y \ : \ J_1 \to \mathrm{R}$ *is defined by*

$$\mathbf{I}_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} y(\zeta) d\zeta, \ \ t > 0,$$

provided that the right hand side is pointwise defined on J_1.

Definition2.2. *[3]. The left sided Riemann-Liouville derivative of order* α $(0 < \alpha < 1)$ *for a function* $y \in \mathbf{L}^1(J_1, \mathrm{R})$ *can be written as*

$$\mathbf{D}_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\zeta)^{-\alpha} y(\zeta) d\zeta = \mathbf{D}\mathbf{I}_{0^+}^{1-\alpha} y(t), \ \ t > 0.$$

Definition2.3. *[3]. The left sided Caputo derivative of order* α $(0 < \alpha < 1)$ *for a function* $y \in \mathbf{C}^1(J_1, \mathrm{R})$ *is defined by*

$${}^c\mathbf{D}_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\zeta)^{-\alpha} y'(\zeta) d\zeta = \mathbf{I}_{0^+}^{1-\alpha} \mathbf{D} y(t), \ \ t > 0.$$

Remark 2.1.

i) *Let* $n-1 < \alpha < n$ $(\alpha \notin \mathrm{N})$. *Then the relationship between the Caputo derivative and the Riemann-Liouville fractional derivative of order* α *is given by*

$${}^c\mathbf{D}_{0^+}^{\alpha} y(t) = \mathbf{D}_{0^+}^{\alpha} y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, \ \ t > 0,$$

where $n = [\alpha] + 1$. *In particular, if* $0 < \alpha < 1$, *then*

$${}^c\mathbf{D}_{0^+}^{\alpha} y(t) = \mathbf{D}_{0^+}^{\alpha} y(t) - \frac{y(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \ \ t > 0.$$

ii) *If* $\alpha = n \in \mathrm{N}$ *and the classical derivative* $y^{(n)}(t)$ *of order* n *exists, then*

$${}^c\mathbf{D}_{0^+}^{\alpha} y(t) = y^{(n)}(t), \ \ t > 0.$$

iii) *The Caputo fractional derivative for every constant function is equal to zero.*

Lemma 2.1. *[3]. Let* $\alpha > 0$ *and* $y \in \mathbf{L}^1(J_1, \mathrm{R})$. *Then* $\mathbf{D}_{0^+}^{\alpha} \mathbf{I}_{0^+}^{\alpha} y(t) = y(t)$,

$t \in J_1$. *Moreover, if* $0 < \alpha < 1$, *then* $\mathbf{I}_{0^+}^{\alpha}\, {}^{c}\mathbf{D}_{0^+}^{\alpha} y(t) = y(t) - y(0),\ t \in J_1$.

Definition2.4. *A function* $y \in \mathbf{G}_4$ *is said to be a solution of the problem (1)-(2) if* y *satisfies the Neutral FFDE* ${}^{c}\mathbf{D}_{0^+}^{\alpha}[y(t) - g(t, y_t)] = f(t, y_t)$, $t \in J_1$, *where* ${}^{c}\mathbf{D}_{0^+}^{\alpha} g(t, y_t)$ *exists, and the following condition* $\beta y_0 - y_b = \phi \in \mathbf{B}$ *holds too.*

Definition2.5. *[20]. A linear topological space of functions from* J_2 *into* R, *with seminorm* $\|\cdot\|_{\mathbf{B}}$, *is called an admissible phase space if* $\mathbf{B}$ *has the following properties:*

(**H1**) *If* $y : J \to R$ *is continuous on* J_1 *and* $y_0 \in \mathbf{B}$, *then for every* $t \in J_1$ *the following conditions hold:*

a. $y_t \in \mathbf{B}$;

b. $|y(t)| \le \mathbf{H}\|y_t\|_{\mathbf{B}}$, *where* $\mathbf{H} > 0$ *is a constant, and* $|\phi(0)| \le \mathbf{H}\|\phi\|_{\mathbf{B}}$ *for all* $\phi \in \mathbf{B}$;

c. $\|y_t\|_{\mathbf{B}} \le K(t) \sup_{0 \le \zeta \le t} |y(\zeta)| + M(t)\|y_0\|_{\mathbf{B}}$, *where* $K, M : [0, +\infty) \to [0, +\infty)$ *with* K *continuous and* M *locally bounded, such that* K, M *are independent of* $y(.)$, *where* $K_b = \sup\{K(t) : t \in J_1\}$ *and* $M_b = \sup\{M(t) : t \in J_1\}$.

(**H2**) *For the function* $y(.)$ *in* (**H1**), *the function* $t \to y_t$ *is continuous from* J_1 *into* $\mathbf{B}$.

(**H3**) *The space* $\mathbf{B}$ *is complete.*

Lemma 2.2. *[21] (Hölder's inequality). Assume that* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in \mathbf{L}^p([a,b], \mathrm{R})$ *and* $g \in \mathbf{L}^q([a,b], \mathrm{R})$. *Then Hölder's inequality for integrals states that*

$$\int_a^b |f(t)g(t)|\,dt \le \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,dt\right)^{\frac{1}{q}}.$$

Main Results

In this section, we are concerned with the existence and uniqueness of solution of the BVP for the Neutral FFDE (1)-(2). The following assumptions will be used in the sequel.

(E1) There exists a $L_f > 0$ such that

$$|f(t,y_1)-f(t,y_2)| \le L_f \|y_1-y_2\|_{\mathbf{B}}, \ t\in J_1,\ y_1,y_2\in \mathbf{B}.$$

(E2) There exists a $L_g > 0$ such that

$$|g(t,y_1)-g(t,y_2)| \le L_g \|y_1-y_2\|_{\mathbf{B}}, \ t\in J_1,\ y_1,y_2\in \mathbf{B}.$$

(E3) The function $f(t,\cdot): \mathbf{B}\to \mathrm{R}$ is continuous for every $t\in J_1$, and for every $x\in\mathbf{B}$, the function $f(\cdot,x): J_1\to\mathrm{R}$ is strongly measurable.

(E4) There exist functions $m\in \mathbf{L}^p(J_1,\mathrm{R})$, $p>\frac{1}{\alpha}$ and a continuously nondecreasing function $\Upsilon:[0,+\infty)\to[0,+\infty)$ such that, for each $t\in J_1$ and $y\in\mathbf{B}$,

$$|f(t,y)| \le m(t)\Upsilon(\|y\|_{\mathbf{B}}).$$

(E5) g is completely continuous and for any bounded set in $\mathbf{G}_4$, the set $\{t\to g(t,y_t): y_t\in\mathbf{B}\}$ is equicontinuous in $C(J_1,\mathrm{R})$ and there exist constants $c_1\in(0,1)$, $c_2\in[0,+\infty)$ such that, for $t\in J_1$ and $y\in\mathbf{B}$

$$|g(t,y)| \le c_1\|y\|_{\mathbf{B}} + c_2.$$

Equivalent integral equation

In this subsection, we need the following auxiliary lemma to prove our results on the problem (1)-(2):

Lemma 3.1. *A solution $y(t)$ of the nonlinear neutral FFDE (1)-(2) on J_1 is given by*

$$y(t)=\begin{cases} g(t,y_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta-\frac{\beta}{\beta-1}g\left(0,\frac{y_b+\phi}{\beta}\right) \\ +\frac{1}{\beta-1}\left(g(b,y_b)+\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta+\phi(0)\right), \ t\in J_1, \\ \frac{1}{\beta}\left(g(b+t,y_{b+t})+\frac{1}{\Gamma(\alpha)}\int_0^{b+t}(b+t-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta\right) \\ \quad -\frac{1}{(\beta-1)}g\left(0,\frac{y_b+\phi}{\beta}\right)+\frac{1}{\beta(\beta-1)}\left(g(b,y_b)\right. \\ \left.+\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta+\phi(0)\right)+\frac{\phi(t)}{\beta}, \ t\in J_2. \end{cases} \quad (3.1)$$

Proof.Firstly, by **(E3)**, it will be easy show that

$$\left|\int_0^t (t-\zeta)^{\alpha-1} f(\zeta, y_\zeta) d\zeta\right| < \infty. \tag{3.2}$$

Applying the operator $\mathbf{I}_{0^+}^{\alpha}$ on both sides of the equation (1), we obtain

$\mathbf{I}_{0^+}^{\alpha}\ {}^{c}\mathbf{D}_{0^+}^{\alpha}\big(y(t)-g(t,y_t)\big)=\mathbf{I}_{0^+}^{\alpha} f(t,y_t).$

In view of Lemma 2.1and Remark 2.1, the solution of equation (1) can be written as

$$\begin{aligned} y(t) &= g(t,y_t)+y(0)-g(0,y_0) \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} f(\zeta,y_\zeta)d\zeta. \end{aligned} \tag{3.3}$$

By the condition (2), we have $\beta y(0)=y(b)+\phi(0),$ it follows from (3.3) that

$$\begin{aligned} y(0) &= \frac{1}{\beta-1}\Big(g(b,y_b)-g(0,y_0) \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^b (b-\zeta)^{\alpha-1} f(\zeta,y_\zeta)d\zeta+\phi(0)\Big). \end{aligned} \tag{3.4}$$

Also, we have

$$y_0=\frac{y_b+\phi}{\beta}. \tag{3.5}$$

From (3.3), (3.4) and (3.5), we get

$$y(t)=\begin{cases} g(t,y_t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} f(\zeta,y_\zeta)d\zeta \\ \quad -\frac{\beta}{\beta-1}g\Big(0,\frac{y_b+\phi}{\beta}\Big)+\frac{1}{\beta-1}\Big(g(b,y_b) \\ \quad +\frac{1}{\Gamma(\alpha)}\int_0^b (b-\zeta)^{\alpha-1} f(\zeta,y_\zeta)d\zeta+\phi(0)\Big),\ t\in J_1 \end{cases} \tag{3.6}$$

Now, since $t\le b,$ if $t\in J_2,$ then $b+t\in J_1$. Thus from (2) and (3.6), we obtain

$$y(t)=\begin{cases} \frac{1}{\beta}\Big(g(b+t,y_{b+t})+\frac{1}{\Gamma(\alpha)}\int_0^{b+t}(b+t-\zeta)^{\alpha-1} f(\zeta,y_\zeta)d\zeta\Big) \\ \quad -\frac{1}{(\beta-1)}g\Big(0,\frac{y_b+\phi}{\beta}\Big)+\frac{\phi(t)}{\beta}+\frac{1}{\beta(\beta-1)}\Big(g(b,y_b) \\ \quad +\frac{1}{\Gamma(\alpha)}\int_0^b (b-\zeta)^{\alpha-1} f(\zeta,y_\zeta)d\zeta+\phi(0)\Big),\ \ t\in J_2 \end{cases} \tag{3.7}$$

From (3.6) and (3.7), the

$$y(t)=\begin{cases} g(t,y_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta-\frac{\beta}{\beta-1}g\left(0,\frac{y_b+\phi}{\beta}\right) \\ +\frac{1}{\beta-1}\left(g(b,y_b)+\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta+\phi(0)\right), \quad t\in J_1, \\ \frac{1}{\beta}\left(g(b+t,y_{b+t})+\frac{1}{\Gamma(\alpha)}\int_0^{b+t}(b+t-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta\right) \\ \quad -\frac{1}{(\beta-1)}g\left(0,\frac{y_b+\phi}{\beta}\right)+\frac{1}{\beta(\beta-1)}\left(g(b,y_b)\right. \\ \quad \left.+\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta+\phi(0)\right)+\frac{\phi(t)}{\beta}, \quad t\in J_2. \end{cases} \tag{3.8}$$

On the other hand, if (3.6) is satisfied, then by definition of $\mathbf{I}_{0^+}^{\alpha}$ and Remark 2.1, for each $t\in J_1$, we have

$$\begin{aligned} &{}^c\mathbf{D}_{0^+}^{\alpha}(y(t)-g(t,y_t)) \\ &= {}^c\mathbf{D}_{0^+}^{\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_0^t(t-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta)\right) \\ &= {}^c\mathbf{D}_{0^+}^{\alpha}\mathbf{I}_{0^+}^{\alpha}f(t,y_t) \\ &= D_{0^+}^{\alpha}\left(\mathbf{I}_{0^+}^{\alpha}f(t,y_t)-\mathbf{I}_{0^+}^{\alpha}f(t,y_t)|_{t=0}\ \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right) \\ &= D_{0^+}^{\alpha}\mathbf{I}_{0^+}^{\alpha}f(t,y_t) \\ &= f(t,y_t). \end{aligned}$$

According to (3.2), we know that $\mathbf{I}_{0^+}^{\alpha}f(t,y_t)|_{t=0}=0$, which means that ${}^c\mathbf{D}_{0^+}^{\alpha}(y(t)-g(t,y_t))=f(t,y_t)$, for $t\in J_1$. Finally, for every $\zeta\in J_2$, then $b+\zeta\in J_1$. Therefore, by (3.6) and (3.7), we get

$$\begin{aligned} \beta y_0(\zeta)-y_b(\zeta) &= \beta y(\zeta)-y(b+\zeta) \\ &= g(b+t,y_{b+t})+\frac{1}{\Gamma(\alpha)}\int_0^{b+t}(b+t-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta \\ &\quad -\frac{\beta}{\beta-1}g\left(0,\frac{y_b+\phi}{\beta}\right)+\frac{1}{(\beta-1)}\left(g(b,y_b)\right. \\ &\quad \left.+\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta+\phi(0)\right)+\phi(\zeta) \\ &\quad -g(b+t,y_{b+t})-\frac{1}{\Gamma(\alpha)}\int_0^{b+t}(b+t-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta \end{aligned}$$

$$+\frac{\beta}{\beta-1}g\left(0,\frac{y_b+\phi}{\beta}\right)-\frac{1}{(\beta-1)}\Big(g(b,y_b)$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta+\phi(0)\Big)$$

$$=\phi(\zeta),$$

i.e. $\beta y_0-y_b=\phi.$ So this completes the proof.

Existence Theorem

In this part, we give the result of existence that depends on the Schauder FPT.

Theorem 3.1 *Assume that hypotheses **(E3)-(E5)** hold. If*

$$\limsup_{\xi\to+\infty}\frac{\Upsilon(\xi)}{\xi}<\frac{1-\left(1+\frac{1}{\beta}\right)c_1K_b\Gamma(\alpha+1)}{b^{\alpha+1}\|m\|_p K_b},\tag{3.9}$$

then there exists at least a solution to problem (1)-(2) on J.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator $\mathbf{N}_4:\mathbf{G}_4\to\mathbf{G}_4$, defined by

$$(\mathbf{N}_4y)(t)=\begin{cases} g(t,y_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta-\frac{\beta}{\beta-1}g\left(0,\frac{y_b+\phi}{\beta}\right) \\ +\frac{1}{\beta-1}\left(g(b,y_b)+\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta+\phi(0)\right), \quad t\in J_1, \\ \frac{1}{\beta}\left(g(b+t,y_{b+t})+\frac{1}{\Gamma(\alpha)}\int_0^{b+t}(b+t-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta\right) \\ -\frac{1}{(\beta-1)}g\left(0,\frac{y_b+\phi}{\beta}\right)+\frac{1}{\beta(\beta-1)}\Big(g(b,y_b) \\ +\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,y_\zeta)d\zeta+\phi(0)\Big)+\frac{\phi(t)}{\beta}, \quad t\in J_2. \end{cases}$$

Let $\tilde{y}:J\to\mathrm{R}$ be the function defined by

$$\tilde{y}(t)=\begin{cases} \frac{\phi(0)}{\beta-1}-\frac{1}{\beta-1}g\left(0,\frac{\tilde{y}_b+\phi}{\beta}\right) \\ +\frac{1}{\beta-1}\left(g(b,\tilde{y}_b)+\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,\tilde{y}_\zeta)d\zeta\right), \quad t\in J_1, \\ \frac{\phi(t)}{\beta}-\frac{1}{\beta-1}g\left(0,\frac{\tilde{y}_b+\phi}{\beta}\right)+\frac{1}{\beta}\Big[g(b+t,\tilde{y}_{b+t}) \\ +\frac{1}{\Gamma(\alpha)}\int_0^{b+t}(b+t-\zeta)^{\alpha-1}f(\zeta,\tilde{y}_\zeta)d\zeta\Big]+\frac{1}{\beta(\beta-1)}\Big[g(b,\tilde{y}_b) \\ +\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}f(\zeta,\tilde{y}_\zeta)d\zeta+\phi(0)\Big], \quad t\in J_2. \end{cases}\tag{3.10}$$

Then

$$\tilde{y}_0 = \frac{1}{\beta-1}[\phi(0) - g\left(0, \frac{\tilde{y}_b + \phi}{\beta}\right)]$$
$$+\frac{1}{\beta-1}\left[g(b,\tilde{y}_b) + \frac{1}{\Gamma(\alpha)}\int_0^b (b-\zeta)^{\alpha-1} f(\zeta,\tilde{y}_\zeta)d\zeta\right].$$

For each $w \in C(J_1, \mathrm{R})$ with $w(0) = 0$, let $\tilde{w} : J \to \mathrm{R}$ be the extension of w to J such that

$$\tilde{w}(t) = \begin{cases} w(t), & t \in J_1, \\ 0, & t \in J_2. \end{cases} \tag{3.11}$$

Then, $\tilde{w}_0 = 0$. If $y(.)$ verifies the integral equation

$$y(t) = \begin{cases} g(t, y_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} f(\zeta, y_\zeta)d\zeta - \frac{\beta}{\beta-1} g\left(0, \frac{y_b+\phi}{\beta}\right) \\ + \frac{1}{\beta-1}\left[g(b, y_b) + \frac{1}{\Gamma(\alpha)}\int_0^b (b-\zeta)^{\alpha-1} f(\zeta, y_\zeta)d\zeta + \phi(0)\right], \ t \in J_1 \end{cases}$$

with $\beta y_0 - y_b = \phi$. By (3.10) and (3.11), we can writing $y(t) = \tilde{y}(t) + \tilde{w}(t)$, for $t \in J_1$, which implies $y_t = \tilde{y}_t + \tilde{w}_t$, for every $t \in J_1$ and the function $w(.)$ satisfies

$$w(t) = g(t, \tilde{y}_t + \tilde{w}_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} f(\zeta, \tilde{y}_\zeta + \tilde{w}_\zeta)d\zeta$$
$$-g\left(0, \frac{\tilde{y}_b + \tilde{w}_b + \phi}{\beta}\right), \ t \text{ in } J_1. \tag{3.12}$$

Moreover, $w_0 = 0$. Set $\mathbf{G}_4^* := \{w \in \mathbf{G}_4 \text{ such that } w_0 = 0\}$. For any $w \in \mathbf{G}_4^*$, we define

$$\|w\|_{\mathbf{G}_4^*} = \|w_0\|_{\mathbf{B}} + \|w\|_C = \sup\{|w(t)| : t \in J_1\}. \tag{3.13}$$

Thus, $(\mathbf{G}_4^*, \|w\|_{\mathbf{G}_4^*})$ is a Banach space. Let $\mathbf{N}_4^* : \mathbf{G}_4^* \to \mathbf{G}_4^*$ be the operator defined by

$$(\mathbf{N}_4^* w)(t) = \begin{cases} g(t, \tilde{y}_t + \tilde{w}_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} f(\zeta, \tilde{y}_\zeta + \tilde{w}_\zeta)d\zeta \\ \quad -g\left(0, \frac{\tilde{y}_b + \tilde{w}_b + \phi}{\beta}\right), \quad t \in J_1, \\ \quad 0, \quad t \in J_2. \end{cases} \tag{3.14}$$

Thus, $\left(\mathbf{N}_4^* w\right)_0 = 0$. It is evident that the operator $\mathbf{N}_4$ has a fixed point is equivalent to $\mathbf{N}_4^*$ that has a fixed point too, and so we go ahead to proving that $\mathbf{N}_4^*$ has a fixed point. By Schauder's FPT, the fixed points of the operator $\mathbf{N}_4^*$ are solutions of the nonlinear neutral FFDE (1)-(2). The proof will be divided into several steps.

Step 1: $\mathbf{N}_4^*$ is continuous. Let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence such that $w_n \to w$ in $\mathbf{G}_4^*$ as $n\to\infty$. Then for each $t \in J_1$, we get

$$\begin{aligned}
&\left|(\mathbf{N}_4^* w_n)(t) - (\mathbf{N}_4^* w)(t)\right| \\
&\le \left|g(t, \tilde{y}_t + (\tilde{w}_n)_t) - g(t, \tilde{y}_t + \tilde{w}_t)\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}\left|f(\zeta, \tilde{y}_\zeta + (\tilde{w}_n)_\zeta) - f(\zeta, \tilde{y}_\zeta + \tilde{w}_\zeta)\right| d\zeta \\
&\quad + \left|g(0, \frac{\tilde{y}_b + (\tilde{w}_n)_b + \phi}{\beta}) - g(0, \frac{\tilde{y}_b + \tilde{w}_b + \phi}{\beta})\right|.
\end{aligned}$$

Thus, from the continuity of f and the complete continuity of g, it follows from the Lebesgue dominated convergence theorem that

$$\left\|\mathbf{N}_4^* w_n - \mathbf{N}_4^* w\right\|_{\mathbf{G}_4^*} \to 0 \text{ as } n \to \infty.$$

Therefore, the operator $\mathbf{N}_4^*$ is continuous.

Step 2: $\mathbf{N}_4^*$ maps bounded subsets into bounded subsets in $\mathbf{G}_4^*$.

Define a ball $B_r = \{w \in \mathbf{G}_4^* : \|w\|_{\mathbf{G}_4^*} \le r\}$ and we prove that for any $r > 0$, there exists a positive constant ℓ such that $\left\|\mathbf{N}_4^* w\right\|_\Omega \le \ell$ for any $w \in B_r$. Indeed, by **(E4)** and **(E5)**, we have for any $w \in B_r$ and for every $t \in J_1$,

$$\begin{aligned}
\left|(\mathbf{N}_4^* w)(t)\right| &\le \left|g(t, \tilde{y}_t + \tilde{w}_t)\right| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}\left|f(\zeta, \tilde{y}_\zeta + \tilde{w}_\zeta)\right| d\zeta \\
&\quad + \left|g(0, \frac{\tilde{y}_b + \tilde{w}_b + \phi}{\beta})\right| \\
&\le \left(c_1\left\|\tilde{y}_t + \tilde{w}_t\right\|_{\mathbf{B}} + c_2\right) \qquad (3.15) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}\Upsilon(\left\|\tilde{y}_\zeta + \tilde{w}_\zeta\right\|_{\mathbf{B}}) m(\zeta) d\zeta \\
&\quad + \left(c_1\left\|\frac{\tilde{y}_b + \tilde{w}_b + \phi}{\beta}\right\|_{\mathbf{B}} + c_2\right).
\end{aligned}$$

Since

$$\sup\left\{\left|\tilde{y}(\tau)\right| \,:\, 0\le\tau\le b\right\}$$

$$\le \frac{\|\phi(0)\|}{\beta-1}+\frac{1}{\beta-1}\left\|g\left(0,\frac{\tilde{y}_b+\phi}{\beta}\right)\right\|+\frac{1}{\beta-1}\|g(b,\tilde{y}_b)\|$$

$$+\frac{1}{\beta-1}\frac{1}{\Gamma(\alpha)}\int_0^b(b-\zeta)^{\alpha-1}\|f(\zeta,\tilde{y}_\zeta)\|d\zeta$$

$$\le \frac{\mathbf{H}\|\phi\|_{\mathbf{B}}}{\beta-1}+\frac{2}{\beta-1}\sup_{(t,\tilde{y})\in J_1\times\mathbf{B}}|g(t,\tilde{y})|$$

$$+\frac{b^\alpha}{(\beta-1)\Gamma(\alpha+1)}\sup_{(t,\tilde{y})\in J_1\times\mathbf{B}}|f(t,\tilde{y})|$$

and

$$\|\tilde{y}_0\|_{\mathbf{B}}\le \frac{\mathbf{H}\|\phi\|_{\mathbf{B}}}{\beta-1}+\frac{2}{\beta-1}\sup_{(t,\tilde{y})\in J_1\times\mathbf{B}}|g(t,\tilde{y})|$$

$$+\frac{b^\alpha}{(\beta-1)\Gamma(\alpha+1)}\sup_{(t,\tilde{y})\in J_1\times\mathbf{B}}|f(t,\tilde{y})|.$$

Set $\frac{\mathbf{H}\|\phi\|_{\mathbf{B}}}{\beta-1}+\frac{2}{\beta-1}\sup\limits_{(t,\tilde{y})\in J_1\times\mathbf{B}}|g(t,\tilde{y})|+\frac{b^\alpha}{(\beta-1)\Gamma(\alpha+1)}\sup\limits_{(t,\tilde{y})\in J_1\times\mathbf{B}}|f(t,\tilde{y})| := K_0.$

Therefore,

$$\|\tilde{y}_t+\tilde{w}_t\|_{\mathbf{B}}$$

$$\le\|\tilde{y}_t\|_{\mathbf{B}}+\|\tilde{w}_t\|_{\mathbf{B}}$$

$$\le K(t)\sup_{0<\tau\le b}\|\tilde{y}(\tau)\|+M(t)\|\tilde{y}_0\|_{\mathbf{B}}+K(t)\sup_{0<\tau\le b}|\tilde{w}(\tau)|+M(t)\|\tilde{w}_0\|_{\mathbf{B}}$$

$$\le K_bK_0+M_bK_0+K_b\sup_{0<\tau\le t}|w(\tau)|$$

$$\le K_0\left(K_b+M_b\right)+K_br \tag{3.16}$$

$$:= r_{0,} \tag{3.17}$$

where $M_b=\sup\{M(t):\, t\in J_1\}$, and also,

$$\left\|\frac{\tilde{y}_b+\tilde{w}_b+\phi}{\beta}\right\|_{\mathbf{B}}\le\frac{1}{\beta}\left(\|\tilde{y}_b\|_{\mathbf{B}}+\|\tilde{w}_b\|_{\mathbf{B}}+\|\phi\|_{\mathbf{B}}\right)$$

$$=\frac{1}{\beta}\left(K_0\left(K_b+M_b\right)+K_br+\|\phi\|_{\mathbf{B}}\right) \tag{3.18}$$

$$:= r_1. \tag{3.19}$$

Now, by (**E2**), (**E3**) and use Hölder's inequality, the (3.15) becomes

$$\begin{aligned}\left|(\mathbf{N}_4^*w)(t)\right| &\le \frac{\Upsilon(r_0)}{\Gamma(\alpha)}\left[\int_0^t (t-\zeta)^{(\alpha-1)q}d\zeta\right]^{\frac{1}{q}}\|m\|_p \\ &\quad +\left(c_1r_0+c_2\right)+\left(c_1r_1+c_2\right) \\ &\le \left(c_1(r_0+r_1)+2c_2\right)+\frac{b^{\alpha+1}\|m\|_p}{\Gamma(\alpha+1)}\Upsilon(r_0) \\ &:= \ell,\end{aligned}$$

where $q>1,\ \frac{1}{p}<\alpha,\ \frac{1}{p}+\frac{1}{q}=1$ and $\|m\|_p=\left(\int_0^t|m(\zeta)|^p d\zeta\right)^{\frac{1}{p}}$. So, $\|\mathbf{N}_4^*w\|_{\mathbf{G}_4^*}\le\ell$. This means that $\mathbf{N}_4^*$ maps bounded subsets into bounded subsets in $\mathbf{G}_4^*$.

Step 3: $\mathbf{N}_4^*$ maps bounded subsets into equicontinuous. Let B_r be a bounded set of $\mathbf{G}_4^*$ as defined in Step 2. Let $w\in B_r$ and $t_1,t_2\in J_1$ with $t_1<t_2$, then we have

$$\begin{aligned}&\left|(\mathbf{N}_4^*w)(t_2)-(\mathbf{N}_4^*w)(t_1)\right| \\ &\le \left|g(t_2,\tilde{y}_{t_2}+\tilde{w}_{t_2})-g(t_1,\tilde{y}_{t_1}+\tilde{w}_{t_1})\right| \\ &\quad +\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}((t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1})f(\zeta,\tilde{y}_\zeta+\tilde{w}_\zeta)d\zeta\right. \\ &\quad \left.+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}((t_2-\zeta)^{\alpha-1}f(\zeta,\tilde{y}_\zeta+\tilde{w}_\zeta)d\zeta\right|\end{aligned}$$

By the complete continuity of g imply that

$\left|g(t_2,\tilde{y}_{t_2}+\tilde{w}_{t_2})-g(t_1,\tilde{y}_{t_1}+\tilde{w}_{t_1})\right|\to 0,$ as $t_2\to t_1$.

It follows from **(E4)**, (3.14), (3.17) and Hölder inequality that

$$\left|(\mathbf{N}_4^* w)(t_2)-(\mathbf{N}_4^* w)(t_1)\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left|(t_2-\zeta)^{\alpha-1}-(t_1-\zeta)^{\alpha-1}\right|\left|f(\zeta,\tilde{y}_\zeta+\tilde{w}_\zeta)\right|d\zeta$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}((t_2-\zeta)^{\alpha-1}\left|f(\zeta,\tilde{y}_\zeta+\tilde{w}_\zeta)\right|d\zeta$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_2}((t_1-\zeta)^{\alpha-1}-(t_2-\zeta)^{\alpha-1})m(\zeta)\Upsilon(\left\|\tilde{y}_\zeta+\tilde{w}_\zeta\right\|_{\mathbf{B}})d\zeta$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-\zeta)^{\alpha-1})m(\zeta)\Upsilon(\left\|\tilde{y}_\zeta+\tilde{w}_\zeta\right\|_{\mathbf{B}})d\zeta$$

$$\leq \frac{\Upsilon(r_0)}{\Gamma(\alpha)}\left[\int_0^{t_1}((t_1-\zeta)^{\alpha-1}-(t_2-\zeta)^{\alpha-1})^q d\zeta\right]^{\frac{1}{q}}\|m\|_p$$

$$+\frac{\Upsilon(r_0)}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}(t_2-\zeta)^{(\alpha-1)q}d\zeta\right]^{\frac{1}{q}}\|m\|_p$$

$$\leq \frac{\Upsilon(r_0)\|m\|_p}{e_2\Gamma(\alpha)}((t_1^{e_1}-t_2^{e_1})+2(t_2-t_1)^{e_1})$$

$$\leq \frac{2\Upsilon(r_0)\|m\|_p}{e_2\Gamma(\alpha)}(t_2-t_1)^{e_1},$$

where r_0 is defined as in Step 2, $e_1=\frac{(\alpha-1)q+1}{q}$, and $e_2=\left((\alpha-1)q+1\right)^{\frac{1}{q}}>0.$ The conclusion is $\left|(\mathbf{N}_4^* w)(t_2)-(\mathbf{N}_4^* w)(t_1)\right|\to 0$ as $t_2-t_1\to 0$, and since w is arbitrary in B_r, this implies that the set $\mathbf{N}_4^* B_r$ is equicontinuous. As consequence of the Arzela--Ascoli theorem with combining steps 1--3, we can conclude that $\mathbf{N}_4^*:\mathbf{G}_4^*\to\mathbf{G}_4^*$ is continuous and completely continuous.

To applying Schauder's FPT, we need to establish that there exists a closed convex subset B_ε in $\mathbf{G}_4^*$ such that $\mathbf{N}_4^* B_\varepsilon\subset B_\varepsilon$. For each positive integer ε, we define $B_\varepsilon=\{w\in\mathbf{G}_4^*:\|w\|_{\mathbf{G}_4^*}\leq\varepsilon\}$, then for each ε, it is easy to verify that B_ε is closed, convex subsets of $\mathbf{G}_4^*$.

We claim that there exists a positive integer ε such that $\mathbf{N}_4^* B_\varepsilon\subset B_\varepsilon$. If this property is false, then for each positive integer ε, there is a function $w_\varepsilon\in B_\varepsilon$ such that $(\mathbf{N}_4^* w_\varepsilon)\notin\mathbf{N}_4^* B_\varepsilon$, then $\left\|\mathbf{N}_4^* w_\varepsilon(t)\right\|_{\mathbf{G}_4^*}>\varepsilon$ for some $t^\varepsilon\in J_1$ where t^ε denotes t depending on ε. However, by use the preceding assumptions we have

$$
\begin{aligned}
\varepsilon &< \left\| \mathbf{N}_4^* w_\varepsilon \right\|_{\mathbf{G}_4^*} \\
&= \sup_{0<t\le b} \left| (\mathbf{N}_4^* w_\varepsilon)(t) \right| \\
&\le \sup_{0<t\le b} \Bigg\{ \left| g(t, \tilde{y}_t + (\tilde{w}_\varepsilon)_t) \right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \left| f(\zeta, \tilde{y}_\zeta + (\tilde{w}_\varepsilon)_\zeta) \right| d\zeta \\
&\quad + \left| g\left(0, \frac{\tilde{y}_b + (\tilde{w}_\varepsilon)_b + \phi}{\beta}\right) \right| \Bigg\} \\
&\le \sup_{0<t\le b} \Bigg\{ \left(c_1 \left\| \tilde{y}_t + (\tilde{w}_\varepsilon)_t \right\|_{\mathbf{B}} + c_2 \right) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} m(\zeta) d\zeta \\
&\quad + \left(c_1 \left\| \frac{\tilde{y}_b + (\tilde{w}_\varepsilon)_b + \phi}{\beta} \right\|_{\mathbf{B}} + c_2 \right) \Bigg\}.
\end{aligned}
$$

From (3.16) we have $\left\| y_t + (\tilde{w}_\varepsilon)_t \right\|_{\mathbf{B}} \le K_0 (K_b + M_b) + K_b \varepsilon := \xi$, which implies $\varepsilon = \frac{\xi}{K_b} - \frac{K_0(K_b+M_b)}{K_b}$. Hence $\left\| \frac{\tilde{y}_b + (\tilde{w}_\varepsilon)_b + \phi}{\beta} \right\|_{\mathbf{B}} \le \frac{1}{\beta}\xi + \frac{1}{\beta}\|\phi\|_{\mathbf{B}}$, it follows from the Hölder inequality that

$$
\begin{aligned}
\xi &< K_0 (K_b + M_b) \\
&\quad + K_b \sup_{0<t\le b} \Bigg\{ (c_1\xi + c_2) + \frac{\Upsilon(\xi)}{\Gamma(\alpha)} \left(\int_0^t (t-\zeta)^{(\alpha-1)q} d\zeta \right)^{\frac{1}{q}} \|m\|_p \\
&\quad + \left(\frac{1}{\beta} c_1 (\xi + \|\phi\|_{\mathbf{B}})) + c_2 \right) \Bigg\} \\
&\le K_b (K_0 + M_b) + K_b \left(c_1 (\xi + \frac{1}{\beta} (\xi + \|\phi\|_{\mathbf{B}}) + 2c_2 \right) \\
&\quad + \frac{b^{\alpha+1} \|m\|_p K_b}{\Gamma(\alpha+1)} \Upsilon(\xi).
\end{aligned}
$$

By dividing ξ on both sides and taking upper limit as $\xi \to +\infty$, we get,

$$
\frac{1 - \left(1 + \frac{1}{\beta}\right) c_1 K_b \Gamma(\alpha+1)}{b^{\alpha+1} \|m\|_p K_b} < \limsup_{\xi\to+\infty} \frac{\Upsilon(\xi)}{\xi}.
$$

This is contrary to (3.9). Hence, for some positive integer ε, we must have $\mathbf{N}_4^* B_\varepsilon \subseteq B_\varepsilon$, i.e. $\mathbf{N}_4^* : B_\varepsilon \to B_\varepsilon$.

An application of Schauder FPT shows that there exists at least a fixed point w of $\mathbf{N}_4^*$ in $\mathbf{G}_4^*$. Thus, $y = \tilde{y} + \tilde{w}$ is a fixed point of $\mathbf{N}_4$ in $\mathbf{G}_4$ which is the solution to (1)-(2) on J, and the proof is completed.

Uniqueness Theorem

In this portion, we give the result of uniqueness that depends on the Banach FPT.

Theorem3.2. *Assume that the conditions **(E1)** and **(E2)** are satisfied. Moreover, if*

$$\left(2L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f\right) K_b < 1, \tag{3.20}$$

where $K_b = \sup\{K(t) : t \in J_1\}$, *then the nonlinear Neutral FFDE (1)-(2) has a unique solution on* J.

Proof. Consider the operator $\mathbf{N}_4^* : \mathbf{G}_4^* \to \mathbf{G}_4^*$ defined by Theorem 3.1. Here, the Banach FPT is concerned with proving that $\mathbf{N}_4^*$ has a fixed point. Since the operator $\mathbf{N}_4^*$ is well defined, it is sufficient to prove that $\mathbf{N}_4^* : \mathbf{G}_4^* \to \mathbf{G}_4^*$ is a contraction mapping. Indeed, in view of **(E1)**, **(E2)** and (3.14), then for each $w, \varpi \in \mathbf{G}_4^*$ and for any $t \in J_1$, we have

$$\begin{aligned}
&\left|(\mathbf{N}_4^* w)(t) - (\mathbf{N}_4^* \varpi)(t)\right| \\
&\le \left|g(t, \tilde{y}_t + \tilde{w}_t) - g(t, \tilde{y}_t + \tilde{\varpi}_t)\right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} \left|f(\zeta, \tilde{y}_\zeta + \tilde{w}_\zeta) - f(\zeta, \tilde{y}_\zeta + \tilde{\varpi}_\zeta)\right| d\zeta \\
&\quad + \left|g\left(0, \frac{\tilde{y}_b + \tilde{w}_b + \phi}{\beta}\right) - g\left(0, \frac{\tilde{y}_b + \tilde{\varpi}_b + \phi}{\beta}\right)\right| \\
&\le L_g \left(\left\|\tilde{w}_t - \tilde{\varpi}_t\right\|_{\mathbf{B}} + \left\|\tilde{w}_b - \tilde{\varpi}_b\right\|_{\mathbf{B}}\right) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} L_f \left\|\tilde{w}_\zeta - \tilde{\varpi}_\zeta\right\|_{\mathbf{B}} d\zeta.
\end{aligned}$$

From **(H1)** and (3.13), for any $t \in J_1$, we have $\left\|\tilde{w}_t - \tilde{\varpi}_t\right\|_{\mathbf{B}} \le K_b \left\|w - \varpi\right\|_{\mathbf{G}_4^*}$. Hence

$$\left|(\mathbf{N}_4^* w)(t) - (\mathbf{N}_4^* \varpi)(t)\right| \le \left(2L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f\right) K_b \left\|w - \varpi\right\|_{\mathbf{G}_4^*}.$$

which implies

$$\left\|\mathbf{N}_4^* w - \mathbf{N}_4^* \varpi\right\|_{\mathbf{G}_4^*} \le \left(2L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f\right) K_b \left\|w - \varpi\right\|_{\mathbf{G}_4^*}.$$

It follows from (3.20) that $\mathbf{N}_4^*$ is a contraction operator. As a consequence of the Banach contraction principle, we can conclude that $\mathbf{N}_4^*$ has a unique fixed point $w \in \mathbf{G}_4^*$ which is just the unique solution to the integral equation (3.12) on J_1. Set $y = \tilde{y} + \tilde{w}$, then $\mathbf{N}_4$ has a unique fixed point $y \in \mathbf{G}_4$ that is the unique solution of the problem (1)-(2) on J.

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