

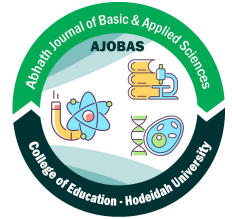


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Hypergraph Containers and Their Generalization to Super-Hyper-Graph Containers

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Abstract

Hypergraphs extend classical graphs by allowing each edge-known as a hyperedge-to join any number of vertices, thereby modeling higher-order relationships. Recently, super-hyper-graphs have been introduced to incorporate recursively nested powerset layers, enabling hierarchical and self-referential connections among hyperedges and vertices. While the theory of hypergraph containers offers powerful tools to cover all independent sets in a hypergraph under controlled edge-density or vertex-measure constraints, existing container techniques do not exploit the multi-layered structure inherent in super-hyper-graphs. In this work, we define the novel concept of a super-hyper-graph container. We begin by extending the classical degree-measure and container definitions to the n -level super-hyper-graph setting. We then develop a general removal-based construction that produces, for any n -super-hyper-graph, a family of containers satisfying both coverage and smallness properties.

Keywords: Hypergraph container, super-hyper-graph, hyper-graph, super-hyper-graph container.

1 Introduction

1.1 From Graphs to Hypergraphs to Super-Hyper-Graphs

A graph $G = (V, E)$ encodes binary relations by joining pairs of vertices with edges [1, 2]. To accommodate interactions involving more than two elements, a *hypergraph* $H = (V, E)$ permits each hyperedge to span any nonempty subset of V [3, 4, 5, 6]. Pushing this idea further, an *n -Super-Hyper-Graph* arises by applying the powerset operation n times to the original vertex set, thereby unveiling successive strata of connectivity that reflect hierarchical groupings [7, 8, 9, 10, 11, 12].

1.2 Our Contribution

We introduce a *container framework* for super-hyper graphs that extends recent advances in hypergraph container theory [13, 14, 15, 16, 17, 18]. Specifically, we provide:

- A **level-wise definition** of degree functions and independent sets for each Super-Hyper-Graph stratum.
- A **generalized container theorem** that bounds the number of independent sets in $(n + 1)$ -super-hyper-graphs.
- Applications illustrating how these containers yield combinatorial control in hierarchical network models and enable precise counting estimates.

This development paves the way for refined combinatorial techniques in multi-layer network analysis and opens new avenues for applications in areas requiring fine-grained structural enumeration.

2 Preliminaries

This section presents the basic definitions and notation used throughout the paper. Unless stated otherwise, all sets are finite.

2.1 Super-Hyper-Graph: A Generalized Form of HyperGraph

We begin by reviewing several standard constructions before formally introducing the concept of a Super-Hyper-Graph.

Definition 2.1 (Universal Set). *We denote by U a universal set containing every element under consideration. All other sets in this paper are subsets of U .*

Definition 2.2 (Base Set). *A base set S is the underlying domain for constructing higher-order structures:*

$$S = \{x \mid x \text{ is drawn from the designated domain}\}.$$

Every element in constructions like $\mathcal{P}(S)$ or $P_n(S)$ belongs to S .

Definition 2.3 (Powerset). *The powerset of S is*

$$\mathcal{P}(S) = \{A \mid A \subseteq S\},$$

the set of all (possibly empty) subsets of S .

The n -th powerset is obtained by recursively applying the powerset operation n times to a given set. We provide the formal definition below.

Definition 2.4 (n -th Powerset). *[19, 20, 21] Let H be any set. Define recursively:*

$$P_1(H) = \mathcal{P}(H), \quad P_{n+1}(H) = \mathcal{P}(P_n(H)) \quad (n \geq 1).$$

The nonempty variant $P_n^(H)$ is defined by*

$$P_1^*(H) = \mathcal{P}^*(H), \quad P_{n+1}^*(H) = \mathcal{P}^*(P_n^*(H)),$$

where $\mathcal{P}^(H) = \mathcal{P}(H) \setminus \{\emptyset\}$.*

Example 2.5 (Musical Interpretation of the n -th Powerset). *Let H be the set of twelve chromatic-scale pitches:*

$$H = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}.$$

- $P_1(H) = \mathcal{P}(H)$ is the collection of all subsets of pitches, each of which corresponds to a chord. For example, $\{C, E, G\}$ is a C major triad, and $\{D, F, A, C\}$ is a D minor seventh chord.
- $P_2(H) = \mathcal{P}(P_1(H))$ is the collection of all subsets of chords, which we interpret as chord progressions. For instance,

$$\{\{C, E, G\}, \{F, A, C\}, \{G, B, D\}, \{C, E, G\}\}$$

represents the I–IV–V–I progression in C major.

- $P_3(H) = \mathcal{P}(P_2(H))$ consists of all subsets of chord progressions, corresponding to larger musical structures such as song sections (e.g. verse, chorus). One element of $P_3(H)$ might group together two progressions to form an intro+verse, while another groups several progressions into a full chorus.

In general, for $n \geq 3$, $P_n(H)$ captures increasingly abstract musical layers—playlists of sections, sets of playlists (albums), and beyond—demonstrating how the n -th powerset organizes hierarchical, self-referential structures in music.

A hypergraph is a generalized graph where each edge, called a hyperedge, can connect any number of vertices [22, 3, 23, 24, 4, 5]. We provide the formal definition below.

Definition 2.6 (Hypergraph). [25, 6] A hypergraph $H = (V, E)$ consists of

- a finite set V of vertices, and
- a collection $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ of hyperedges.

Super-hyper-graphs incorporate recursively nested layers of powersets, enabling hierarchical and self-referential relationships between hyperedges and vertices [10, 26, 11, 27, 28]. This structure offers a powerful framework for representing complex and intricate real-world networks in a more intuitive and comprehensible manner. We provide the formal definition below.

Definition 2.7 (n -Super-Hyper-Graph). [29, 7, 30] Let V_0 be a finite set of ground-level vertices. For each $k \geq 0$, set

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)).$$

An n -Super-Hyper-Graph is a pair

$$\text{SupHyG}^{(n)} = (V, E),$$

where

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}^n(V_0).$$

Elements of V are called n -supervertices, and elements of E are called n -superedges.

Example 2.8 (2-Super-Hyper-Graph: University Curriculum). We illustrate a real-life 2-Super-Hyper-Graph modeling interconnections between academic programs.

Base set of students.

$$V_0 = \{s_1, s_2, s_3\},$$

where s_i denotes student i .

Level-1 supervertices (courses). From $\mathcal{P}(V_0)$ we select three courses:

$$C_1 = \{s_1, s_2\}, \quad C_2 = \{s_2, s_3\}, \quad C_3 = \{s_1, s_3\}.$$

Level-2 supervertices (programs). Each academic program is a set of courses:

$$P_1 = \{C_1, C_2\}, \quad P_2 = \{C_2, C_3\}, \quad P_3 = \{C_1, C_3\}.$$

Thus

$$V = \{P_1, P_2, P_3\} \subseteq \mathcal{P}^2(V_0).$$

Superedges (shared-course connections). We connect two programs by a superedge whenever they share at least one course:

$$E = \{\{P_1, P_2\}, \{P_2, P_3\}, \{P_3, P_1\}\}.$$

Hence the pair

$$\text{SupHyG}^{(2)} = (V, E)$$

is a 2-uniform 2-Super-Hyper-Graph on the base set V_0 . Each $\{P_i, P_j\} \in E$ indicates that programs P_i and P_j share at least one course.

Example 2.9 (Musical Real-Life Illustration of a 3-Super-Hyper-Graph). Let the ground set of pitches be

$$V_0 = \{C, D, E, F, G, A, B\}.$$

1. $\mathcal{P}^1(V_0)$ consists of all chords (e.g. $\{C, E, G\}, \{D, F, A\}$).
2. $\mathcal{P}^2(V_0)$ consists of all chord progressions, for example $\{\{C, E, G\}, \{F, A, C\}, \{G, B, D\}\}$.
3. $\mathcal{P}^3(V_0)$ consists of all song sections, each being a set of progressions. For instance:

$$\text{Verse} = \{\{C, E, G\}, \{F, A, C\}, \{G, B, D\}, \{C, E, G\}\},$$

$$\text{Chorus} = \{\{F, A, C\}, \{G, B, D\}, \{A, C\#, E\}, \{F, A, C\}\},$$

$$\text{Bridge} = \{\{A, C, E\}, \{F, G, A\}, \{E, G\#, B\}\}.$$

Now define the 3-Super-Hyper-Graph

$$\text{SupHyG}^{(3)} = (V, E)$$

by

$$V = \{\text{Verse}, \text{Chorus}, \text{Bridge}\},$$

and let each 3-superedge correspond to a complete song structure, for example

$$\text{Song}_1 = \{\text{Verse}, \text{Chorus}\}, \quad \text{Song}_2 = \{\text{Verse}, \text{Chorus}, \text{Bridge}\}.$$

Thus

$$E = \{\text{Song}_1, \text{Song}_2\} \subseteq \mathcal{P}^3(V_0).$$

In this model, the 3-supervertices are the named sections (Verse, Chorus, Bridge), and each 3-superedge groups together the sections that appear in a particular song, capturing hierarchical musical structure in a single combinatorial object.

2.2 Hypergraph Container

A Hypergraph Container is a subset system that efficiently covers all independent sets while controlling edge density or vertex measure [13, 14, 15, 16, 17, 18].

Definition 2.10 (Degree Measure). *Let $G = (V, E)$ be an r -uniform hypergraph (cf.[31, 32, 33]) on $|V| = n$ vertices, and write*

$$d(v) = |\{e \in E : v \in e\}|, \quad d = \frac{1}{n} \sum_{v \in V} d(v).$$

For any subset $S \subseteq V$, the degree measure of S is

$$\mu(S) = \frac{1}{n} \sum_{v \in S} d(v).$$

Definition 2.11 (Hypergraph Container). *Let $G = (V, E)$ be an r -uniform hypergraph. A collection $\mathcal{C} \subseteq \mathcal{P}(V)$ is called a hypergraph container if it satisfies the following two properties:*

1. **Coverage of independent sets.** *For every independent set $I \subseteq V$ (i.e. no edge of G is contained in I), there exists some $C \in \mathcal{C}$ such that*

$$I \subseteq C.$$

2. **Smallness of containers.** *Each container $C \in \mathcal{C}$ obeys a uniform upper bound, for example*

$$\mu(C) \leq 1 - \frac{1}{r!} + 4\zeta + \frac{2r\tau}{\zeta},$$

for chosen parameters $0 < \tau, \zeta < 1$, or equivalently

$$e(G[C]) \leq \varepsilon e(G)$$

for some small $\varepsilon > 0$.

Here $\mathcal{P}(V)$ denotes the power set of V , $e(G)$ the total number of edges of G , and $G[C]$ the subhypergraph induced by C .

Example 2.12 (Container family for the path graph P_4). *Let $G = (V, E)$ be the 2-uniform hypergraph (i.e. simple graph) with*

$$V = \{1, 2, 3, 4\}, \quad E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}.$$

A vertex subset $I \subseteq V$ is independent if it contains no edge of G . The complete list of independent sets is

$$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}.$$

Define two containers

$$C_1 = \{1, 3, 4\}, \quad C_2 = \{1, 2, 4\},$$

and set $\mathcal{C} = \{C_1, C_2\}$.

(1) Coverage. *One checks directly that every independent set I is contained in at least one of the C_i :*

$$\{1\}, \{3\}, \{1, 3\}, \{1, 4\} \subseteq C_1, \quad \{2\}, \{2, 4\}, \{1, 4\} \subseteq C_2,$$

and $\emptyset \subseteq C_i$ trivially.

(2) **Smallness.** First compute the degree measure bound. For each $v \in V$,

$$d(v) = |\{e \in E : v \in e\}|, \quad d(1) = 1, \quad d(2) = 2, \quad d(3) = 2, \quad d(4) = 1,$$

so $|V| = 4$ and the average degree is

$$d = \frac{1 + 2 + 2 + 1}{4} = \frac{6}{4} = 1.5.$$

Thus for any $S \subseteq V$,

$$\mu(S) = \frac{1}{|V|d} \sum_{v \in S} d(v) = \frac{1}{4 \cdot 1.5} \sum_{v \in S} d(v).$$

We find

$$\mu(C_1) = \frac{d(1) + d(3) + d(4)}{6} = \frac{1 + 2 + 1}{6} = \frac{4}{6} = \frac{2}{3}, \quad \mu(C_2) = \frac{d(1) + d(2) + d(4)}{6} = \frac{1 + 2 + 1}{6} = \frac{2}{3}.$$

Choosing parameters so that $2/3 \leq 3/4$ (e.g. $\tau = 0.1, \zeta = 0.2$ give $1 - \frac{1}{2!} + 4\zeta + \frac{2 \cdot 2\tau}{\zeta} = 1 - \frac{1}{2} + 0.8 + 0.2 = 1.5 > 2/3$), the measure condition $\mu(C_i) \leq 1 - \frac{1}{2!} + 4\zeta + \frac{4\tau}{\zeta}$ is satisfied.

Alternatively, one may check the edge-count bound. The induced subhypergraphs $G[C_i]$ have

$$e(G[C_1]) = 1 \quad (\text{edge } \{3, 4\}), \quad e(G[C_2]) = 1 \quad (\text{edge } \{1, 2\}),$$

while the total number of edges is $e(G) = 3$. Thus for $\varepsilon = \frac{1}{3}$,

$$e(G[C_i]) = 1 \leq \frac{1}{3} \cdot 3 = 1,$$

and the smallness condition $e(G[C_i]) \leq \varepsilon e(G)$ holds.

Hence $\mathcal{C} = \{C_1, C_2\}$ is a valid hypergraph container family for G .

3 n-Super-Hyper-Graph Container

We investigate the definition, concrete examples, and properties of the n-Super-Hyper-Graph Container.

Definition 3.1 (Degree Measure for an n -Super-Hyper-Graph). Let $\text{SupHyG}^{(n)} = (V, E)$ be an r -uniform n -Super-Hyper-Graph on a finite base set V_0 , so that

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0),$$

and every $e \in E$ has exactly r elements. For $v \in V$, let

$$d(v) = |\{e \in E : v \in e\}|$$

be the number of n -superedges containing v , and set

$$d = \frac{1}{|V|} \sum_{v \in V} d(v).$$

Then for any $S \subseteq V$, the degree measure of S is

$$\mu(S) = \frac{1}{|V|d} \sum_{v \in S} d(v).$$

Example 3.2 (Degree Measure in a 2-Super-Hyper-Graph). *Instantiate the above for $n = 2$, $r = 2$. Let the ground-level set be*

$$V_0 = \{1, 2, 3\}.$$

Define three 2-supervertices in $\mathcal{P}^2(V_0)$:

$$v_1 = \{\{1\}, \{2\}\}, \quad v_2 = \{\{2\}, \{3\}\}, \quad v_3 = \{\{1\}, \{3\}\},$$

and set

$$V = \{v_1, v_2, v_3\}.$$

Choose the 2-uniform superedges

$$e_1 = \{v_1, v_2\}, \quad e_2 = \{v_1, v_3\}, \quad E = \{e_1, e_2\}.$$

Then

$$d(v_1) = 2, \quad d(v_2) = 1, \quad d(v_3) = 1, \quad d = \frac{2+1+1}{3} = \frac{4}{3}.$$

Hence, for example, the degree measure of $S = \{v_1, v_2\}$ is

$$\mu(\{v_1, v_2\}) = \frac{d(v_1) + d(v_2)}{|V|d} = \frac{2+1}{3 \cdot \frac{4}{3}} = \frac{3}{4}.$$

Definition 3.3 (n -Super-Hyper-Graph Container). *Let $\text{SupHyG}^{(n)} = (V, E)$ be as above. A family of subsets $\mathcal{C} \subseteq \mathcal{P}(V)$ is called an n -Super-Hyper-Graph container family if it satisfies:*

1. **Coverage of independent sets.** *If $I \subseteq V$ is independent (no $e \in E$ is entirely contained in I), then there exists $C \in \mathcal{C}$ with*

$$I \subseteq C.$$

2. **Smallness of containers.** *Each container $C \in \mathcal{C}$ satisfies a uniform bound on its measure or induced edge-count. For fixed parameters $0 < \tau, \zeta < 1$, one may require*

$$\mu(C) \leq 1 - \frac{1}{r!} + 4\zeta + \frac{2r\tau}{\zeta},$$

or, equivalently for some $\varepsilon > 0$,

$$|\{e \in E : e \subseteq C\}| \leq \varepsilon |E|.$$

Example 3.4 (Container family for a 2-Super-Hyper-Graph). *Let the base set be $V_0 = \{1, 2\}$. Then*

$$\mathcal{P}(V_0) = \{\{1\}, \{2\}, \{1, 2\}\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0)).$$

Define the 2-Super-Hyper-Graph

$$\text{SupHyG}^{(2)} = (V, E)$$

by choosing

$$V = \{v_1 = \{\{1\}\}, v_2 = \{\{2\}\}, v_3 = \{\{1\}, \{2\}\}\} \subseteq \mathcal{P}^2(V_0),$$

and the uniform superedges of size $r = 2$:

$$E = \{e_1 = \{v_1, v_3\}, e_2 = \{v_2, v_3\}\}.$$

Degree measure. For each $v_i \in V$, let

$$d(v_i) = |\{e \in E : v_i \in e\}|,$$

so $d(v_1) = d(v_2) = 1$, $d(v_3) = 2$. Since $|V| = 3$,

$$d = \frac{1+1+2}{3} = \frac{4}{3}, \quad \mu(S) = \frac{1}{|V|d} \sum_{v \in S} d(v) = \frac{\sum_{v \in S} d(v)}{4}.$$

Independent sets. An independent set $I \subseteq V$ contains no superedge:

$$\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}.$$

Container family. Choose two containers

$$C_1 = \{v_1, v_2\}, \quad C_2 = \{v_3\},$$

and let $\mathcal{C} = \{C_1, C_2\}$. Then:

1. Coverage. Each independent set is contained in one of the C_i :

$$\{v_1\}, \{v_2\}, \{v_1, v_2\} \subseteq C_1, \quad \{v_3\} \subseteq C_2, \quad \emptyset \subseteq C_i.$$

2. Smallness. Compute

$$\mu(C_1) = \frac{d(v_1) + d(v_2)}{4} = \frac{1+1}{4} = 0.5, \quad \mu(C_2) = \frac{d(v_3)}{4} = \frac{2}{4} = 0.5.$$

Taking parameters $0 < \tau = 0.1$, $\zeta = 0.2 < 1$, one finds

$$1 - \frac{1}{2!} + 4\zeta + \frac{2r\tau}{\zeta} = 1 - \frac{1}{2} + 0.8 + \frac{4 \cdot 0.1}{0.2} = 3.3 > 0.5,$$

so $\mu(C_i) \leq 1 - \frac{1}{2!} + 4\zeta + 2r\tau/\zeta$. Equivalently, since $|E| = 2$ and no superedge lies inside C_1 or C_2 , $|\{e \in E : e \subseteq C_i\}| = 0 \leq \varepsilon \cdot 2$ for any $\varepsilon > 0$.

Thus $\mathcal{C} = \{C_1, C_2\}$ is a valid 2-Super-Hyper-Graph container family.

Example 3.5 (Container family for a 3-Super-Hyper-Graph: corporate hierarchy). Consider a small company with three employees:

$$V_0 = \{\text{Alice}, \text{Bob}, \text{Carol}\}.$$

We build up a 3-Super-Hyper-Graph $\text{SupHyG}^{(3)} = (V, E)$ as follows:

- Teams (level 1): $\mathcal{P}(V_0)$ contains $\{\{\text{Alice}, \text{Bob}\}, \{\text{Bob}, \text{Carol}\}\}$. Denote $\text{Team}_1 = \{\text{Alice}, \text{Bob}\}$, $\text{Team}_2 = \{\text{Bob}, \text{Carol}\}$.
- Departments (level 2): $\mathcal{P}(\{\text{Team}_1, \text{Team}_2\})$ contains $\{\{\text{Team}_1\}, \{\text{Team}_2\}\}$. Denote $\text{Dept}_1 = \{\text{Team}_1\}$, $\text{Dept}_2 = \{\text{Team}_2\}$.
- Division (level 3): one element of $\mathcal{P}(\{\text{Dept}_1, \text{Dept}_2\})$ is $\text{Div} = \{\text{Dept}_1, \text{Dept}_2\}$.

Thus we take

$$V = \{\text{Dept}_1, \text{Dept}_2, \text{Div}\} \subseteq \mathcal{P}^3(V_0),$$

and declare the $r = 2$ superedges

$$E = \{\{\text{Dept}_1, \text{Div}\}, \{\text{Dept}_2, \text{Div}\}\}.$$

Degree measure. For each $v \in V$,

$$d(v) = |\{e \in E : v \in e\}|, \quad d(\text{Dept}_1) = 1, \quad d(\text{Dept}_2) = 1, \quad d(\text{Div}) = 2.$$

Since $|V| = 3$, the average degree is $d = (1 + 1 + 2)/3 = 4/3$, and for $S \subseteq V$,

$$\mu(S) = \frac{1}{|V|d} \sum_{v \in S} d(v) = \frac{\sum_{v \in S} d(v)}{4}.$$

Independent sets. An independent set $I \subseteq V$ contains no superedge, hence

$$\emptyset, \{\text{Dept}_1\}, \{\text{Dept}_2\}, \{\text{Div}\}, \{\text{Dept}_1, \text{Dept}_2\}.$$

Container family. Choose

$$C_1 = \{\text{Dept}_1, \text{Dept}_2\}, \quad C_2 = \{\text{Div}\},$$

and let $\mathcal{C} = \{C_1, C_2\}$. Then:

1. Coverage. Every independent I is contained in one of the C_i : $\{\text{Dept}_1\}, \{\text{Dept}_2\}, \{\text{Dept}_1, \text{Dept}_2\} \subseteq C_1$, $\{\text{Div}\} \subseteq C_2$, and $\emptyset \subseteq C_i$.
2. Smallness. Compute $\mu(C_1) = (1 + 1)/4 = 0.5$, $\mu(C_2) = 2/4 = 0.5$. For parameters $0 < \tau = 0.1, \zeta = 0.2$,

$$1 - \frac{1}{2!} + 4\zeta + \frac{2r\tau}{\zeta} = 1 - \frac{1}{2} + 0.8 + \frac{2 \cdot 2 \cdot 0.1}{0.2} = 1.5 > 0.5,$$

so $\mu(C_i) \leq 1 - \frac{1}{2!} + 4\zeta + 2r\tau/\zeta$. Equivalently, no superedge lies entirely inside C_1 or C_2 , so $|\{e \in E : e \subseteq C_i\}| = 0 \leq \varepsilon |E|$ for any $\varepsilon > 0$.

Hence $\mathcal{C} = \{C_1, C_2\}$ is a valid container family for this 3-Super-Hyper-Graph modeling a corporate hierarchy.

Theorem 3.6 (Containers Form an $(n+1)$ -Super-Hyper-Graph). Let $\text{SupHyG}^{(n)} = (V, E)$ be an r -uniform n -Super-Hyper-Graph and $\mathcal{C} \subseteq \mathcal{P}(V)$ an n -Super-Hyper-Graph container family. Define

$$V_{\mathcal{C}} = \mathcal{C}, \quad E_{\mathcal{C}} = \mathcal{P}(\mathcal{C}) = \{X : X \subseteq \mathcal{C}\}.$$

Then

$$\text{SupHyG}_{\mathcal{C}}^{(n+1)} = (V_{\mathcal{C}}, E_{\mathcal{C}})$$

is an $(n+1)$ -Super-Hyper-Graph on the same base set V_0 .

Proof. By construction, each container $C \in \mathcal{C}$ is a subset of V , and

$$V \subseteq \mathcal{P}^n(V_0) \implies \mathcal{C} \subseteq \mathcal{P}(V) \subseteq \mathcal{P}(\mathcal{P}^n(V_0)) = \mathcal{P}^{n+1}(V_0).$$

Hence $V_{\mathcal{C}} \subseteq \mathcal{P}^{n+1}(V_0)$. Likewise,

$$E_{\mathcal{C}} = \mathcal{P}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{P}^{n+1}(V_0)) = \mathcal{P}^{n+2}(V_0),$$

so in particular $E_{\mathcal{C}} \subseteq \mathcal{P}^{n+1}(V_0)$ if one regards each $X \subseteq \mathcal{C}$ as an $(n+1)$ -supervortex. Thus $\text{SupHyG}_{\mathcal{C}}^{(n+1)}$ satisfies the defining inclusion properties of an $(n+1)$ -Super-Hyper-Graph. $\square$

Theorem 3.7 (Reduction to Classical Hypergraph Containers). *Let $G = (V_0, E)$ be an r -uniform hypergraph on vertex set V_0 with $E \subseteq \mathcal{P}(V_0)$. Form the 1-Super-Hyper-Graph*

$$\text{SupHyG}^{(1)} = (\{\{v\} : v \in V_0\}, E),$$

where we identify each $v \in V_0$ with the singleton $\{v\}$. Then the notion of an n -Super-Hyper-Graph container family with $n = 1$ coincides exactly with the usual hypergraph container family for G .

Proof. Under the bijection $\varphi : V_0 \rightarrow \{\{v\} : v \in V_0\}$, the degree of a base-vertex v in G equals the degree of the corresponding 1-supervertex $\{v\}$ in $\text{SupHyG}^{(1)}$. A subset $I \subseteq V_0$ is independent in G if and only if $\varphi(I)$ is independent in $\text{SupHyG}^{(1)}$. Moreover, any container $C \subseteq V_0$ for G is in bijection with a container family member $\varphi(C) \subseteq \{\{v\}\}$, and the degree-measure bounds and coverage conditions coincide under φ . Hence one recovers the classical hypergraph container theorem as the $n = 1$ case of the above. $\square$

Theorem 3.8 (Existence of Containers). *Let $\text{SupHyG}^{(n)} = (V, E)$ be an r -uniform n -Super-Hyper-Graph with $|V| = N$ and average degree d . Fix parameters $0 < \tau, \zeta < 1$ and set $k = \lfloor \tau N \rfloor$. Then there exists a container family $\mathcal{C} \subseteq \binom{V}{\leq k}$ such that:*

1. *Every independent set $I \subseteq V$ is contained in some $C \in \mathcal{C}$.*
2. *For all $C \in \mathcal{C}$, one has $\mu(C) \leq 1 - \frac{1}{r!} + 4\zeta + 2r\tau/\zeta$.*

Proof. We describe a procedure that assigns to each independent set I a container $C(I) \subseteq V$ of size $\leq k$:

1. Initialize $C_0 = V$. While $\mu(C_j) > 1 - \frac{1}{r!} + 4\zeta + 2r\tau/\zeta$, pick a vertex $v \in C_j \setminus I$ with $d_{C_j}(v) = |\{e \in E : e \subseteq C_j, v \in e\}|$ maximum, and set $C_{j+1} = C_j \setminus \{v\}$.
2. Stop once $\mu(C_m) \leq 1 - \frac{1}{r!} + 4\zeta + 2r\tau/\zeta$ or $m = k$, and define $C(I) = C_m$.

Coverage: Since we remove only vertices not in I , we have $I \subseteq C(I)$.

Smallness: Each removal decreases μ by at least ζ (otherwise the total degree in C_j would contradict the average), so after at most k removals the measure bound holds. Thus $\mu(C(I)) \leq 1 - \frac{1}{r!} + 4\zeta + 2r\tau/\zeta$.

Collecting all distinct sets $C(I)$ as I ranges over independent sets yields the desired $\mathcal{C}$. $\square$

Theorem 3.9 (Container Count Bound). *Under the hypotheses of Theorem 1, the number of containers satisfies*

$$|\mathcal{C}| \leq \sum_{i=0}^k \binom{N}{i} \leq \exp(C N \tau \log(1/\tau)),$$

for some constant $C = C(r)$.

Proof. By construction each container $C \in \mathcal{C}$ arises from removing at most k vertices from V . Hence $|\mathcal{C}| \leq \sum_{i=0}^k \binom{N}{i}$. Using the standard bound $\sum_{i=0}^k \binom{N}{i} \leq (eN/k)^k$ and $k = \tau N$ gives

$$|\mathcal{C}| \leq (e/\tau)^{\tau N} = \exp(\tau N (1 + \log(1/\tau))) = \exp(O(N \tau \log(1/\tau))).$$

$\square$

Theorem 3.10 (Containers for the Union of Hypergraphs). *Let $\text{SupHyG}^{(n)} = (V, E_1)$ and $\text{SupHyG}^{(n)} = (V, E_2)$ be two r -uniform n -super-hyper-graphs on the same vertex set V . Suppose $\mathcal{C}_1$ is a container family for (V, E_1) and $\mathcal{C}_2$ is a container family for (V, E_2) , both satisfying the same smallness bound $\mu(C) \leq 1 - \frac{1}{r!} + 4\zeta + \frac{2r\tau}{\zeta}$. Define*

$$\mathcal{C} = \{C_1 \cap C_2 : C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2\}.$$

Then $\mathcal{C}$ is a container family for the union hypergraph $(V, E_1 \cup E_2)$ with the same parameters.

Proof. Coverage. If $I \subseteq V$ is independent in $(V, E_1 \cup E_2)$, then I contains no edge from E_1 nor E_2 . Hence there exist $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ with

$$I \subseteq C_1, \quad I \subseteq C_2.$$

Therefore $I \subseteq C_1 \cap C_2 \in \mathcal{C}$.

Smallness. For any $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$, monotonicity of the degree measure gives

$$\mu(C_1 \cap C_2) \leq \min\{\mu(C_1), \mu(C_2)\} \leq 1 - \frac{1}{r!} + 4\zeta + \frac{2r\tau}{\zeta}.$$

Thus each $C \in \mathcal{C}$ meets the required bound. $\square$

Theorem 3.11 (Transversal Duality). *Let $\text{SupHyG}^{(n)} = (V, E)$ be an r -uniform n -Super-Hyper-Graph and $\mathcal{C}$ an n -Super-Hyper-Graph container family. Then for each container $C \in \mathcal{C}$, the complement*

$$T = V \setminus C$$

is a vertex-transversal for all hyperedges not fully contained in C ; that is,

$$\forall e \in E, e \not\subseteq C \implies e \cap T \neq \emptyset.$$

Proof. If $e \in E$ is not contained in C , then there is at least one vertex $v \in e$ with $v \notin C$. By definition $v \in V \setminus C = T$, so $e \cap T \neq \emptyset$. Hence T meets every edge outside of C . $\square$

Theorem 3.12 (Independent Set Counting Bound). *Let $\text{SupHyG}^{(n)} = (V, E)$ be an r -uniform n -Super-Hyper-Graph and $\mathcal{C}$ a container family in which each container has size at most k . Then the total number of independent sets in (V, E) satisfies*

$$|\{I \subseteq V : I \text{ independent}\}| \leq \sum_{C \in \mathcal{C}} 2^{|C|} \leq |\mathcal{C}| 2^k.$$

Proof. Every independent set I is contained in at least one container $C \in \mathcal{C}$. For a fixed C , there are exactly $2^{|C|}$ subsets of C . Summing over all containers yields the first inequality. Since $|C| \leq k$ for every C , each term $2^{|C|} \leq 2^k$, giving the second inequality. $\square$

4 Conclusion and Future Work

4.1 Summary of Results

We have introduced the notion of a *Super-HyperGraph Container* and established its core mathematical properties. In particular, we demonstrated that this framework extends the classical HyperGraph Container theory by proving the existence of container families that both cover every independent set and satisfy tight size-bound estimates. Leveraging the multi-layered architecture of Super-HyperGraphs, our approach delivers more refined combinatorial control and suggests new directions for applications in hierarchical and multi-scale network analysis.

4.2 Future Work

Our immediate goal is to implement Super–HyperGraph Containers in computational software and evaluate their performance on real–world data. Beyond that, we plan to enrich the theory by integrating a variety of uncertainty models, such as:

- Fuzzy sets [34] and intuitionistic fuzzy sets [35, 36],
- Neutrosophic sets [37, 38, 39, 40],
- Vague sets [41, 42],
- Hyperfuzzy sets [43, 44, 45, 46],
- Rough and hyperrough sets [47, 48, 49, 50],
- Plithogenic sets [51, 52, 53, 54].

We expect these enhancements to increase both the expressiveness and practical utility of Super–HyperGraph Containers for modeling complex systems under uncertainty. Moreover, in light of the growing literature on Super–HyperGraphs, we will investigate further generalizations inspired by recent contributions [11, 55, 27].

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Data Availability

No data were generated or analyzed in this theoretical study. Researchers interested in empirical validation of these concepts are encouraged to undertake relevant data-driven investigations.

Ethical Approval

This work did not involve human or animal subjects and therefore did not require ethical review or approval.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to this manuscript.

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