



Abhath Journal of Basic and Applied Sciences, 4(1) (2025), 73–93

Abhath Journal of Basic and Applied Sciences

Website: <https://ojs.abhath-ye.com/index.php/OJSRJBAS/about>

Research paper, Short communication, Review, Technical paper



Neutrosophic Matroids, Antimatroids, and Greedoids: A Unified Framework for Combinatorial Structures under Uncertainty

Takaaki Fujita^{1,*}

¹Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan.

*Corresponding author: E-mail: Takaaki.fujita060@gmail.com

Received: 31 May 2025 Accepted: 23 June 2025 Published: 30 June 2025

Abstract

A matroid is a combinatorial structure that extends the concept of linear independence from vector spaces to arbitrary sets. Matroids are widely applied in fields such as discrete optimization due to their rich mathematical properties and their applicability in areas like graph theory and computer science. Antimatroids and greedoids are well-known related concepts. To address uncertainty, the concept of Fuzzy Matroids was developed, extending matroid theory using fuzzy sets and serving as a subject of extensive study. This paper presents the mathematical definitions of Neutrosophic Matroids, Neutrosophic Antimatroids, and Neutrosophic Greedoids. These combinatorial structures extend classical concepts by incorporating Neutrosophic sets, facilitating more advanced modeling of uncertainty and indeterminacy.

Keywords: Neutrosophic set, Fuzzy set, Matroid, Antimatroid, Greedoid.

1 Introduction

1.1 Matroid Theory

A *matroid* is a combinatorial structure that generalizes the concept of linear independence from vector spaces to arbitrary finite sets [1]. Matroids exhibit a rich theoretical framework and have found widespread applications in discrete optimization, graph theory, and computer science [2, 3, 4].

Over time, several generalizations of matroids have been proposed. An *antimatroid* is defined via a family of accessible sets: if a set is feasible, then removing any of its elements one at a time still results in a feasible set [5, 6]. A *greedoid*, on the other hand, is characterized by hereditary and exchange properties on feasible sets, allowing greedy algorithms to produce optimal solutions in various contexts [7, 8, 9, 10]. Due to their foundational importance, numerous extensions and related structures have been explored in the literature [11, 12, 13, 14, 15, 16].

1.2 Models for Uncertainty

Various mathematical frameworks have been developed to address uncertainty and imprecision in real-world problems [17, 18]. Among these, *fuzzy sets* and *neutrosophic sets* have garnered considerable attention.

A fuzzy set assigns to each element a membership degree within the interval $[0, 1]$, thereby capturing the notion of partial membership [19, 20, 21, 22, 23, 24]. In contrast, a neutrosophic set assigns three independent membership values—truth, indeterminacy, and falsity—to each element, with each value taken from $[0, 1]$. This allows for a more nuanced and flexible representation of uncertainty, incompleteness, and inconsistency [25, 26, 27, 28].

Several related models have also been introduced, including *intuitionistic fuzzy sets* [29, 30, 31, 32], *vague sets* [33, 34], *hyperfuzzy sets* [35, 36, 37, 38, 39], and *quadri-partitioned neutrosophic sets* [40, 41]. These models provide additional flexibility in capturing different forms of uncertainty.

Such uncertainty representations have been successfully incorporated into various algebraic and combinatorial structures. One notable example is the *fuzzy matroid*, which generalizes classical matroids by replacing crisp notions of independence with membership functions defined on the ground set [42, 43, 44, 45, 46, 47, 48, 49]. Fuzzy matroids preserve many key properties of matroids while enabling the modeling of imprecise or partial information.

1.3 Contributions of This Paper

In this paper, we introduce the formal definitions of *Neutrosophic Matroids*, *Neutrosophic Antimatroids*, and *Neutrosophic Greedoids*. By incorporating single-valued neutrosophic sets into the classical combinatorial framework, these new structures capture uncertainty, indeterminacy, and inconsistency in a unified manner. In doing so, we extend existing combinatorial theories—matroids, antimatroids, and greedoids—into the neutrosophic domain, laying the groundwork for more flexible models in situations where traditional crisp or fuzzy approaches are insufficient.

2 Basic Concepts in Matroid Theory

In this section, we review the fundamental definitions and essential properties of matroids. For foundational concepts in set theory, the reader is referred to [50]. Throughout this paper, we consider only finite structures. Readers who wish to explore operational or algorithmic details are encouraged to consult the relevant references as needed.

2.1 Matroid

The definition of a matroid is provided below.

Definition 2.1 (Matroid [1]). *Let E be a finite ground set, and let $\mathcal{I} \subseteq \mathcal{P}(E)$ be a collection of subsets of E , called the independent sets. The pair*

$$M = (E, \mathcal{I})$$

is a matroid if the following axioms hold:

1. **Nonempty and Hereditary:** $\emptyset \in \mathcal{I}$, and if $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
2. **Exchange Property:** For any $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, there exists an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The collection $\mathcal{I}$ is called the family of independent sets of M .

Matroids arise in many combinatorial optimization problems, including the matroid intersection problem [51], the matroid optimization problem [52], and the matroid partitioning problem [53]. Derived concepts such as dual matroids [54], polymatroids [9], and graphic matroids [55] are also well studied. Because of its broad range of applications, matroid theory remains an active area of research [56].

Definition 2.2 (Rank Function and Submodularity). *Let $M = (E, \mathcal{I})$ be a matroid. The rank function*

$$r_M : \mathcal{P}(E) \longrightarrow \{0, 1, 2, \dots, |E|\}$$

is defined, for each subset $X \subseteq E$, by

$$r_M(X) = \max\{|I| : I \subseteq X, I \in \mathcal{I}\}.$$

The value $r_M(X)$ is called the rank of X .

A function $\mu : \mathcal{P}(E) \rightarrow [0, \infty)$ is submodular if for all $X, Y \subseteq E$,

$$\mu(X) + \mu(Y) \geq \mu(X \cup Y) + \mu(X \cap Y).$$

Example 2.3 (Graphic Matroid). *Let $G = (V, E)$ be the undirected graph with*

$$V = \{1, 2, 3\}, \quad E = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}.$$

We construct the cycle matroid $M(G)$ on the ground set E as follows:

- **Ground Set:** $E = \{e_1, e_2, e_3\}$, where

$$e_1 = \{1, 2\}, \quad e_2 = \{2, 3\}, \quad e_3 = \{3, 1\}.$$

- **Independent Sets:**

$$\mathcal{I} = \left\{ I \subseteq E : I \text{ contains no cycle in } G \right\}.$$

Concretely, the independent subsets are

$$\begin{aligned} &\emptyset, \\ &\{e_1\}, \{e_2\}, \{e_3\}, \\ &\{e_1, e_2\}, \{e_2, e_3\}, \{e_3, e_1\}. \end{aligned}$$

Note that any two distinct edges do not form a cycle, but $\{e_1, e_2, e_3\}$ is not independent because it is the 3-cycle $\{1, 2, 3\}$.

- **Bases:** *A basis is a maximal independent set, so*

$$\mathcal{B} = \left\{ \{e_1, e_2\}, \{e_2, e_3\}, \{e_3, e_1\} \right\}.$$

Each basis has size 2, which agrees with the rank of $M(G)$.

- **Circuits:** *A circuit is a minimal dependent set. Since $\{e_1, e_2, e_3\}$ forms the unique cycle, the only circuit is*

$$C = \{e_1, e_2, e_3\}.$$

- **Rank Function:** *For any $X \subseteq E$, the rank $r_{M(G)}(X)$ is the size of the largest acyclic subset of X . For instance,*

$$r_{M(G)}(\{e_1, e_2, e_3\}) = 2, \quad r_{M(G)}(\{e_1, e_2\}) = 2, \quad r_{M(G)}(\{e_1\}) = 1.$$

Verification of Matroid Axioms:

1. Hereditary Property: Any subset of an acyclic edge-set remains acyclic. For example, $\{e_1, e_2\}$ is independent, and each of its subsets $\{e_1\}, \{e_2\}, \emptyset$ is also independent.
2. Exchange Property: Suppose $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$. The only case with $|I_1| = 1$ and $|I_2| = 2$ occurs, for instance, when $I_1 = \{e_1\}$ and $I_2 = \{e_2, e_3\}$. There must be some edge $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\}$ is acyclic. Indeed, e_2 or e_3 can be added to $\{e_1\}$ without creating a cycle; e.g. $\{e_1, e_2\} \in \mathcal{I}$.

Thus $M(G) = (E, \mathcal{I})$ is a valid matroid, called the graphic matroid (or cycle matroid) of G .

In matroid theory, the rank function r_M is itself submodular. Related parameters, such as rank-width and its variants, are studied in [57].

2.2 Antimatroids and Greedoids

Next, we recall two other well-studied set systems: antimatroids and greedoids.

Definition 2.4 (Antimatroid [58]). Let E be a finite set and let $\mathcal{F} \subseteq \mathcal{P}(E)$ be a family of subsets. The pair

$$(E, \mathcal{F})$$

is an antimatroid if the following conditions hold:

1. Hereditary Property: If $F \in \mathcal{F}$ and $F' \subseteq F$, then $F' \in \mathcal{F}$.
2. Accessibility Property: If $F \in \mathcal{F}$ and $F \neq \emptyset$, then there exists an element $e \in F$ such that $F \setminus \{e\} \in \mathcal{F}$.
3. Union Closure: If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cup F_2 \in \mathcal{F}$.

The collection $\mathcal{F}$ is called the family of feasible sets. These three axioms characterize the structure of an antimatroid.

Example 2.5 (Antimatroid from a Partial Order). Let $E = \{1, 2, 3\}$, and impose a partial order on E given by

$$1 < 2, \quad 1 < 3,$$

so that element 1 is a prerequisite for both 2 and 3. We construct an antimatroid $(E, \mathcal{F})$ by declaring a subset $F \subseteq E$ to be feasible precisely when F is an order ideal (i.e., a downward-closed subset) in this poset and every nonempty feasible set can remove a maximal element and remain feasible. Concretely, the family of feasible sets is:

$$\mathcal{F} = \{ \emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\} \}.$$

Below we verify that $\mathcal{F}$ satisfies all antimatroid axioms.

• **Ground Set:** $E = \{1, 2, 3\}$.

• **Feasible Sets:**

$$\mathcal{F} = \{ \emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\} \}.$$

Note that any feasible set must contain 1 before it contains 2 or 3, since 1 precedes both 2 and 3 in the partial order.

Verification of Antimatroid Axioms:

1. Hereditary Property: If $F \in \mathcal{F}$ and $F' \subseteq F$, then $F' \in \mathcal{F}$. Indeed:

- $\emptyset \subseteq \{1\}$, and $\emptyset \in \mathcal{F}$.
- $\{1\} \subseteq \{1, 2\}$, and $\{1\} \in \mathcal{F}$.
- $\{1\} \subseteq \{1, 3\}$, and $\{1\} \in \mathcal{F}$.
- $\{1, 2\} \subseteq \{1, 2, 3\}$, and $\{1, 2\} \in \mathcal{F}$.
- $\{1, 3\} \subseteq \{1, 2, 3\}$, and $\{1, 3\} \in \mathcal{F}$.
- Every other pair of subsets either trivially contains $\emptyset$ or is itself in $\mathcal{F}$.

Hence every subset of a feasible set remains feasible.

2. Accessibility Property: If $F \in \mathcal{F}$ and $F \neq \emptyset$, then there exists $e \in F$ such that $F \setminus \{e\} \in \mathcal{F}$. Check each nonempty feasible set:

- $F = \{1\}$: Removing $e = 1$ yields $\emptyset \in \mathcal{F}$.
- $F = \{1, 2\}$: The maximal element in the poset restricted to $\{1, 2\}$ is 2. Removing 2 yields $\{1\} \in \mathcal{F}$.
- $F = \{1, 3\}$: The maximal element in $\{1, 3\}$ is 3. Removing 3 yields $\{1\} \in \mathcal{F}$.
- $F = \{1, 2, 3\}$: Maximal elements are 2 and 3. Removing 2 gives $\{1, 3\} \in \mathcal{F}$, and removing 3 gives $\{1, 2\} \in \mathcal{F}$.

In each case, at least one maximal element can be removed while preserving feasibility.

3. Union Closure: If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cup F_2 \in \mathcal{F}$. Consider any two feasible sets:

- $\{1\} \cup \{1, 2\} = \{1, 2\} \in \mathcal{F}$.
- $\{1\} \cup \{1, 3\} = \{1, 3\} \in \mathcal{F}$.
- $\{1, 2\} \cup \{1, 3\} = \{1, 2, 3\} \in \mathcal{F}$.
- $\{1, 2\} \cup \{1, 2\} = \{1, 2\} \in \mathcal{F}$ (trivially).
- $\emptyset \cup F = F \in \mathcal{F}$ for any $F \in \mathcal{F}$.

All unions of feasible sets remain feasible, since each union still respects the requirement that 1 appear before 2 or 3.

Therefore, $(E, \mathcal{F})$ satisfies all three antimatroid axioms and is called the poset antimatroid induced by the ordering $1 < 2$ and $1 < 3$.

Definition 2.6 (Greedoid [7]). Let E be a finite set and let $\mathcal{F} \subseteq \mathcal{P}(E)$ be a family of subsets. The pair

$$(E, \mathcal{F})$$

is a greedoid if the following conditions are satisfied:

1. $\emptyset \in \mathcal{F}$.
2. (Accessibility) If $F \in \mathcal{F}$ and $F \neq \emptyset$, then there exists an element $e \in F$ such that $F \setminus \{e\} \in \mathcal{F}$.

3. (Exchange-like Property) If $F_1, F_2 \in \mathcal{F}$ with $|F_1| < |F_2|$, then there exists an element $e \in F_2 \setminus F_1$ such that $F_1 \cup \{e\} \in \mathcal{F}$.

Example 2.7 (Branching Greedoid on a Rooted Tree). Consider the directed rooted tree T with root r and vertex set

$$V = \{r, a, b, c\},$$

and with edge set E consisting of three directed edges (all oriented away from r):

$$e_1 : r \longrightarrow a, \quad e_2 : r \longrightarrow b, \quad e_3 : b \longrightarrow c.$$

We define a set system $\mathcal{F} \subseteq \mathcal{P}(E)$ by declaring a subset $F \subseteq E$ to be feasible if and only if the subdigraph induced by F is a (directed) arborescence rooted at r whose vertices are all reachable from r via edges in F . Concretely, the feasible sets are:

$$\mathcal{F} = \left\{ \emptyset, \right. \\ \left. \{e_1\}, \{e_2\}, \right. \\ \left. \{e_2, e_3\}, \{e_1, e_2\}, \right. \\ \left. \{e_1, e_2, e_3\} \right\}.$$

We claim that $(E, \mathcal{F})$ is a greedoid. We check each axiom below:

1. $\emptyset \in \mathcal{F}$. The empty set induces only the root r , which is trivially a valid arborescence.
2. Accessibility: If $F \in \mathcal{F}$ is nonempty, then there exists an edge $e \in F$ whose removal leaves a smaller feasible set. Indeed:
 - If $F = \{e_1\}$ or $F = \{e_2\}$, then removing that single edge yields $\emptyset \in \mathcal{F}$.
 - If $F = \{e_2, e_3\}$, then e_3 is a leaf-edge (its head c has no outgoing edges in F), so $F \setminus \{e_3\} = \{e_2\} \in \mathcal{F}$.
 - If $F = \{e_1, e_2\}$, then either e_1 or e_2 may be removed. For instance, $F \setminus \{e_1\} = \{e_2\} \in \mathcal{F}$.
 - If $F = \{e_1, e_2, e_3\}$, then e_3 is a leaf-edge since its head c has no outgoing edges in F . Removing e_3 yields $\{e_1, e_2\} \in \mathcal{F}$.

In each case, one can remove a leaf-edge (an edge whose head vertex has outdegree zero within F), and the resulting set remains an arborescence rooted at r .

3. Exchange-like Property: Let $F_1, F_2 \in \mathcal{F}$ satisfy $|F_1| < |F_2|$. We must show there is an edge $e \in F_2 \setminus F_1$ such that $F_1 \cup \{e\} \in \mathcal{F}$. The only nontrivial case arises when $|F_1| = 1$ and $|F_2| = 2$, or $|F_1| = 2$ and $|F_2| = 3$. We check each:

- If $F_1 = \{e_1\}$ and $F_2 = \{e_2, e_3\}$, then $F_2 \setminus F_1 = \{e_2, e_3\}$. Adding e_2 to F_1 gives $\{e_1, e_2\}$, which is an arborescence rooted at r ; hence $\{e_1, e_2\} \in \mathcal{F}$.
- If $F_1 = \{e_2\}$ and $F_2 = \{e_1, e_2\}$, then $F_2 \setminus F_1 = \{e_1\}$. Adding e_1 to F_1 yields $\{e_1, e_2\} \in \mathcal{F}$.
- If $F_1 = \{e_2, e_3\}$ and $F_2 = \{e_1, e_2, e_3\}$, then $F_2 \setminus F_1 = \{e_1\}$. Adding e_1 to F_1 gives $\{e_1, e_2, e_3\} \in \mathcal{F}$.
- If $F_1 = \{e_1, e_2\}$ and $F_2 = \{e_1, e_2, e_3\}$, then $F_2 \setminus F_1 = \{e_3\}$. Adding e_3 to F_1 yields $\{e_1, e_2, e_3\} \in \mathcal{F}$.

In every situation, there is at least one edge $e \in F_2 \setminus F_1$ whose addition to F_1 produces a feasible set. Hence the exchange-like property holds.

Therefore, $(E, \mathcal{F})$ satisfies all axioms of a greedoid. This particular structure is known as the branching greedoid of the rooted tree T .

3 Review: Fuzzy Matroids, Fuzzy Antimatroids, and Fuzzy Greedoids

In this section, we present the definitions of fuzzy matroids, fuzzy antimatroids, and fuzzy greedoids.

3.1 Fuzzy Matroid

A *fuzzy matroid* extends the classical notion of a matroid by using fuzzy set theory. Like ordinary matroids, fuzzy matroids have been studied extensively (see, e.g., [42, 49]). We begin by recalling the definition of a fuzzy set.

Definition 3.1 (Fuzzy Set [19, 24]). *Let Y be a nonempty universe. A fuzzy set τ on Y is a function*

$$\tau: Y \longrightarrow [0, 1].$$

A fuzzy relation on Y is a fuzzy subset $\delta \subseteq Y \times Y$ (i.e., a function $\delta: Y \times Y \rightarrow [0, 1]$). If τ is a fuzzy set on Y and δ is a fuzzy relation on Y , then δ is said to be a fuzzy relation on τ provided

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Next, we give the formal definition of a fuzzy matroid.

Definition 3.2 (Fuzzy Matroid [42, 49]). *Let X be a finite ground set. Denote by $F(X)$ the collection of all fuzzy subsets of X ; that is,*

$$F(X) = \{ \tau \mid \tau: X \rightarrow [0, 1] \}.$$

For two fuzzy sets $\tau_1, \tau_2 \in F(X)$, we write $\tau_2 \subset \tau_1$ if

$$\tau_2(x) < \tau_1(x) \quad \text{for every } x \in X.$$

Define the support of a fuzzy set τ by

$$\text{supp}(\tau) = \{ x \in X : \tau(x) > 0 \}.$$

For any $\nu \in F(X)$, let

$$m(\nu) = \min\{\nu(x) : x \in \text{supp}(\nu)\}.$$

A pair $M = (X, \mathcal{I})$, where $\mathcal{I} \subseteq F(X)$, is called a fuzzy matroid if $\mathcal{I}$ satisfies:

1. *Fuzzy Hereditary Property: If $\tau_1 \in \mathcal{I}$ and $\tau_2 \subset \tau_1$, then $\tau_2 \in \mathcal{I}$.*
2. *Fuzzy Exchange Property: Whenever $\tau_1, \tau_2 \in \mathcal{I}$ satisfy $|\text{supp}(\tau_1)| < |\text{supp}(\tau_2)|$, there exists $\tau_3 \in \mathcal{I}$ such that:*

(a) $\tau_1 \subset \tau_3 \subseteq \tau_1 \vee \tau_2$, where

$$(\tau_1 \vee \tau_2)(x) = \max\{\tau_1(x), \tau_2(x)\} \quad \text{for all } x \in X,$$

$$(b) \ m(\tau_3) \geq \min\{m(\tau_1), m(\tau_2)\}.$$

The set $\mathcal{I}$ is called the family of independent fuzzy sets of the fuzzy matroid $M = (X, \mathcal{I})$.

Example 3.3 (A Simple Fuzzy Matroid). Let $X = \{a, b\}$. Define four fuzzy subsets of X as follows:

$$\begin{aligned}\tau_\emptyset(a) &= 0, & \tau_\emptyset(b) &= 0, \\ \tau_a(a) &= 0.7, & \tau_a(b) &= 0, \\ \tau_b(a) &= 0, & \tau_b(b) &= 0.8, \\ \tau_{ab}(a) &= 0.7, & \tau_{ab}(b) &= 0.8.\end{aligned}$$

Let

$$\mathcal{I} = \{\tau_\emptyset, \tau_a, \tau_b, \tau_{ab}\}.$$

Then $(X, \mathcal{I})$ is a fuzzy matroid:

- **Fuzzy Hereditary:** Every strict sub-fuzzy-set of each $\tau \in \mathcal{I}$ is again in $\mathcal{I}$. For instance, $\tau_\emptyset \subset \tau_a$ and $\tau_\emptyset \subset \tau_b$, and $\tau_a, \tau_b \subset \tau_{ab}$, all of which lie in $\mathcal{I}$.
- **Fuzzy Exchange:** Whenever $|\tau_1| < |\tau_2|$, one can choose $\tau_3 = \tau_{ab}$ or $\tau_3 = \tau_a$ (or τ_b) so that

$$\tau_1 \subset \tau_3 \subseteq \tau_1 \vee \tau_2, \quad m(\tau_3) \geq \min\{m(\tau_1), m(\tau_2)\}.$$

For example, between τ_a (support size 1) and τ_{ab} (size 2), we have $\tau_a \subset \tau_{ab} \subseteq \tau_a \vee \tau_{ab}$ and $m(\tau_{ab}) = \min\{0.7, 0.8\} = 0.7 \geq \min\{0.7, 0.7\}$.

Thus $\mathcal{I}$ satisfies both axioms and defines a fuzzy matroid.

3.2 Fuzzy Antimatroids and Fuzzy Greedoids

We now present definitions of fuzzy antimatroids and fuzzy greedoids, which generalize antimatroids and greedoids to the fuzzy setting.

Definition 3.4 (Fuzzy Antimatroid [59, 60]). Let E be a finite ground set, and let $\mathcal{F}_f \subseteq [0, 1]^E$ be a family of fuzzy subsets of E . For $\mu \in [0, 1]^E$, denote its support by

$$\text{supp}(\mu) = \{x \in E : \mu(x) > 0\}.$$

Define the deletion of an element $x \in E$ from μ by

$$\mu_x(y) = \begin{cases} \mu(y), & y \neq x, \\ 0, & y = x. \end{cases}$$

The pair $A = (E, \mathcal{F}_f)$ is a fuzzy antimatroid if:

1. **Fuzzy Deletion Property:** For every $\mu \in \mathcal{F}_f$ with $\text{supp}(\mu) \neq \emptyset$, there exists $x \in \text{supp}(\mu)$ such that

$$\mu_x \in \mathcal{F}_f.$$

2. **Fuzzy Union Closure:** For any $\mu_1, \mu_2 \in \mathcal{F}_f$, their pointwise maximum $\mu_1 \vee \mu_2$, defined by

$$(\mu_1 \vee \mu_2)(y) = \max\{\mu_1(y), \mu_2(y)\} \quad \text{for all } y \in E,$$

also belongs to $\mathcal{F}_f$.

Here, $\mathcal{F}_f$ is called the family of feasible fuzzy sets.

Example 3.5 (A Simple Fuzzy Antimatroid). Let $E = \{x, y\}$ and choose a minimum membership threshold $\tau = 0.5$. Define three fuzzy sets on E :

$$\begin{aligned}\mu_\emptyset(x) &= 0, & \mu_\emptyset(y) &= 0, \\ \mu_x(x) &= 0.7, & \mu_x(y) &= 0, \\ \mu_{xy}(x) &= 0.7, & \mu_{xy}(y) &= 0.6.\end{aligned}$$

Let

$$\mathcal{F}_f = \{\mu_\emptyset, \mu_x, \mu_{xy}\}.$$

We claim $(E, \mathcal{F}_f)$ is a fuzzy antimatroid:

- Fuzzy Deletion Property:

$$(\mu_{xy})_y = \mu_x \in \mathcal{F}_f, \quad (\mu_x)_x = \mu_\emptyset \in \mathcal{F}_f.$$

Thus every nonempty fuzzy set in $\mathcal{F}_f$ admits a deletion that remains feasible.

- Fuzzy Union Closure:

$$\mu_x \vee \mu_{xy} = (x \mapsto \max\{0.7, 0.7\}, y \mapsto \max\{0, 0.6\}) = \mu_{xy} \in \mathcal{F}_f,$$

and similarly for the other pairs.

Since both axioms hold, $(E, \mathcal{F}_f)$ is indeed a fuzzy antimatroid modeling the requirement that y can only be “added” when x already has sufficient membership.

Definition 3.6 (Fuzzy Greedoid [60, 61]). Let E be a finite ground set, and let $\mathcal{F}_f \subseteq [0, 1]^E$ be a collection of fuzzy subsets. For $\mu \in [0, 1]^E$, define its support $\text{supp}(\mu)$ and deletion μ_x as above. Also write $\mu_1 \leq \mu_2$ to mean $\mu_1(x) \leq \mu_2(x)$ for all $x \in E$. The pair

$$G = (E, \mathcal{F}_f)$$

is a fuzzy greedoid if:

1. Deletion Property: For every $\mu \in \mathcal{F}_f$ with $\text{supp}(\mu) \neq \emptyset$, there exists $x \in \text{supp}(\mu)$ such that $\mu_x \in \mathcal{F}_f$.
2. Expansion Property: If $\mu_1, \mu_2 \in \mathcal{F}_f$ satisfy $\text{supp}(\mu_1) \subseteq \text{supp}(\mu_2)$ and $\mu_1 \leq \mu_2$, then there exists $\mu_3 \in \mathcal{F}_f$ such that:
 - (a) $\mu_1 \leq \mu_3 \leq \mu_1 \vee \mu_2$, where $\mu_1 \vee \mu_2$ is defined pointwise by $(\mu_1 \vee \mu_2)(y) = \max\{\mu_1(y), \mu_2(y)\}$ for all $y \in E$,
 - (b) $m(\mu_3) \geq \min\{m(\mu_1), m(\mu_2)\}$, where $m(\nu) = \min\{\nu(x) : x \in \text{supp}(\nu)\}$.

The collection $\mathcal{F}_f$ is then called the family of feasible fuzzy sets of the fuzzy greedoid G .

Example 3.7 (A Simple Fuzzy Greedoid). Let $E = \{e_1, e_2\}$ represent two tasks where e_2 can only follow e_1 . Define three fuzzy sets on E :

$$\begin{aligned}\mu_\emptyset(e_1) &= 0, & \mu_\emptyset(e_2) &= 0, \\ \mu_1(e_1) &= 0.8, & \mu_1(e_2) &= 0, \\ \mu_{12}(e_1) &= 0.8, & \mu_{12}(e_2) &= 0.7.\end{aligned}$$

Set

$$\mathcal{F}_f = \{\mu_\emptyset, \mu_1, \mu_{12}\}.$$

We show $(E, \mathcal{F}_f)$ is a fuzzy greedoid:

- Deletion Property:

$$(\mu_{12})_{e_2} = \mu_1 \in \mathcal{F}_f, \quad (\mu_1)_{e_1} = \mu_\emptyset \in \mathcal{F}_f.$$

Every nonempty fuzzy set admits a deletion that remains in $\mathcal{F}_f$.

- Expansion Property:

- From $\mu_\emptyset \leq \mu_1$ with $(\mu_\emptyset) = \emptyset \subseteq \{e_1\} = (\mu_1)$, take $\mu_3 = \mu_1$. Then $\mu_\emptyset \leq \mu_1 \leq \mu_\emptyset \vee \mu_1 = \mu_1$, and $m(\mu_1) = 0.8 \geq 0$.
- From $\mu_1 \leq \mu_{12}$ with $\{e_1\} \subseteq \{e_1, e_2\}$, take $\mu_3 = \mu_{12}$. Then $\mu_1 \leq \mu_{12} \leq \mu_1 \vee \mu_{12} = \mu_{12}$, and $m(\mu_{12}) = \min\{0.8, 0.7\} = 0.7 \geq \min\{0.8, 0.7\}$.

Thus both deletion and expansion axioms hold, and $\mathcal{F}_f$ defines a fuzzy greedoid.

4 Main Results: Neutrosophic Matroid, Antimatroid, and Greedoid

One of the main results of this paper is the investigation of neutrosophic matroids, antimatroids, and greedoids. Fuzzy combinatorial structures use single-valued membership degrees to model partial inclusion, whereas neutrosophic versions assign independent truth, indeterminacy, and falsity values, enabling more nuanced representation of uncertainty and inconsistency.

4.1 Neutrosophic Matroid

A *neutrosophic matroid* extends the classical matroid concept by employing single-valued neutrosophic sets.

Definition 4.1 (Single-Valued Neutrosophic Set). [62, 63, 64, 65, 66] Let E be a finite, nonempty ground set. A single-valued neutrosophic set (SVNS) μ on E is a collection of ordered triples

$$\mu = \{ \langle e, T_\mu(e), I_\mu(e), F_\mu(e) \rangle : e \in E \},$$

where for each $e \in E$,

$$T_\mu(e), I_\mu(e), F_\mu(e) \in [0, 1].$$

Here:

- $T_\mu(e)$ is the truth-membership degree of e ,
- $I_\mu(e)$ is the indeterminacy-membership degree of e ,
- $F_\mu(e)$ is the falsity-membership degree of e .

There is no restriction on the sum $T_\mu(e) + I_\mu(e) + F_\mu(e)$ other than

$$0 \leq T_\mu(e) + I_\mu(e) + F_\mu(e) \leq 3.$$

Example 4.2 (Customer Satisfaction as a Single-Valued Neutrosophic Set). Consider a survey of 100 customers evaluating three aspects of an online service:

$$E = \{\text{Quality}, \text{Price}, \text{Support}\}.$$

The raw responses were:

- **Quality:** 75 customers satisfied, 10 uncertain, 15 dissatisfied;

- **Price:** 50 customers satisfied, 25 uncertain, 25 dissatisfied;
- **Support:** 60 customers satisfied, 20 uncertain, 20 dissatisfied.

We convert these counts into neutrosophic membership degrees by dividing by 100:

$$\mu = \{ \langle \text{Quality}, 0.75, 0.10, 0.15 \rangle, \langle \text{Price}, 0.50, 0.25, 0.25 \rangle, \langle \text{Support}, 0.60, 0.20, 0.20 \rangle \}.$$

Here:

- $T_\mu(e)$ is the fraction of customers expressing satisfaction with aspect e ;
- $I_\mu(e)$ is the fraction expressing uncertainty;
- $F_\mu(e)$ is the fraction expressing dissatisfaction.

This single-valued neutrosophic set thus models not only how many customers were positive, but also how much uncertainty and negativity existed for each aspect.

Definition 4.3 (Neutrosophic matroid). Let E be a finite set, and let each element $e \in E$ carry a triple

$$(T(e), I(e), F(e)) \in [0, 1]^3$$

with no restriction on $T(e) + I(e) + F(e)$ beyond $0 \leq T, I, F \leq 1$. Fix a reliability threshold $\tau \in (0, 1]$. A neutrosophic matroid is a pair

$$M = (E, \Psi),$$

where Ψ is a nonempty collection of such triples (single-valued neutrosophic sets) satisfying:

- **Hereditary:** If $\mu \in \Psi$ and ν is any neutrosophic set with

$$T_\nu(e) \leq T_\mu(e), \quad I_\nu(e) \geq I_\mu(e), \quad F_\nu(e) \geq F_\mu(e) \quad \forall e \in E,$$

then $\nu \in \Psi$.

- **Exchange:** If $\mu, \nu \in \Psi$ and $|(\mu)| < |(\nu)|$, then there exists $\omega \in \Psi$ and an element $e \in (\nu) \setminus (\mu)$ such that

$$\begin{aligned} (\omega) &= (\mu) \cup \{e\}, \\ T_\omega(f) &\geq \tau \quad \text{for all } f \in (\omega). \end{aligned}$$

Each $\mu \in \Psi$ is called an independent neutrosophic set.

Definition 4.4 (Circuit and Rank). Let $M = (E, \Psi)$ be a neutrosophic matroid.

- A neutrosophic circuit is a minimal $\mu \notin \Psi$ for which every $\nu < \mu$ (in the neutrosophic order) lies in Ψ .
- The rank of any neutrosophic set ν is

$$r_M(\nu) = \max\{ |(\omega)| : \omega \in \Psi, \omega \leq \nu \}.$$

Example 4.5 (Simple Reliability Matroid). Let $E = \{a, b, c\}$, and assign

$$T(a) = 0.8, \quad T(b) = 0.6, \quad T(c) = 0.7,$$

together with $I(e) = 0.1$, $F(e) = 1 - T(e) - 0.1$ for each e . Choose threshold $\tau = 0.6$, and define

$$\Psi = \left\{ \mu : \min_{e \in (\mu)} T(e) \geq \tau \text{ and } |(\mu)| \leq 2 \right\}.$$

Then

$$\Psi = \{ \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\} \},$$

and one checks easily:

- Any sub-set of an allowed support remains allowed (hereditary).
- Given two supports of different sizes, one can add a link with $T \geq \tau$ to the smaller without violating $\min T \geq \tau$ (exchange).

Hence (E, Ψ) is a neutrosophic matroid modeling “choose up to two links, each with reliability ≥ 0.6 ,” with circuits exactly those triples that violate the size or reliability condition.

4.2 Neutrosophic Antimatroid and Neutrosophic Greedoid

As with matroids, antimatroids and greedoids can be lifted to the neutrosophic setting by replacing each crisp subset with a single-valued neutrosophic set.

Definition 4.6 (Neutrosophic Antimatroid). Let E be a finite set, and let $\mathcal{F}_n$ be a family of single-valued neutrosophic sets on E . Fix a reliability threshold $\tau \in (0, 1]$. We say $(E, \mathcal{F}_n)$ is a neutrosophic antimatroid if every $\mu \in \mathcal{F}_n$ satisfies:

(NA1) Deletion: If $\mu \neq \emptyset$, then there is some $x \in (\mu)$ such that

$$\mu_x(e) = \begin{cases} (0, 0, 0), & e = x, \\ (T_\mu(e), I_\mu(e), F_\mu(e)), & e \neq x \end{cases}$$

also lies in $\mathcal{F}_n$.

(NA2) Union Closure: If $\mu_1, \mu_2 \in \mathcal{F}_n$, then their pointwise join

$$(\mu_1 \vee \mu_2)(e) = (\max\{T_{\mu_1}(e), T_{\mu_2}(e)\}, \min\{I_{\mu_1}(e), I_{\mu_2}(e)\}, \min\{F_{\mu_1}(e), F_{\mu_2}(e)\})$$

also belongs to $\mathcal{F}_n$.

In practice, one often takes

$$\mathcal{F}_n = \{ \mu : T_\mu(e) \geq \tau \forall e \in (\mu) \}$$

so that $\mathcal{F}_n$ consists of every neutrosophic set whose support elements all exceed the reliability threshold τ .

Example 4.7 (Antimatroid of Assembled Components). Suppose $E = \{A, B, C\}$ are three parts of a device, with measured reliabilities

$$T(A) = 0.9, \quad T(B) = 0.8, \quad T(C) = 0.7,$$

and indeterminacy $I(e) = 0.1$, falsity $F(e) = 1 - T(e) - 0.1$. Choose $\tau = 0.75$. Then

$$\mathcal{F}_n = \{ \mu_S : S \subseteq E, T(e) \geq 0.75 \text{ for all } e \in S \}.$$

Explicitly,

$$\mathcal{F}_n = \{ \emptyset, \{A\}, \{B\}, \{A, B\} \}.$$

- Deletion: Removing any one assembled part from $\{A, B\}$ leaves $\{A\}$ or $\{B\}$, both still reliable.
- Union Closure: Joining $\{A\}$ and $\{B\}$ yields $\{A, B\}$, which remains in $\mathcal{F}_n$.

Thus $(E, \mathcal{F}_n)$ is a neutrosophic antimatroid modeling progressive assembly under reliability constraints.

Definition 4.8 (Neutrosophic Greedoid). Let E be a finite set and $\mathcal{G}_n$ a family of neutrosophic sets on E . With the same threshold τ , $(E, \mathcal{G}_n)$ is a neutrosophic greedoid if:

(NG1) Deletion: As in (NA1), any nonempty $\mu \in \mathcal{G}_n$ admits a deletion $\mu_x \in \mathcal{G}_n$.

(NG2) Expansion: If $\mu_1, \mu_2 \in \mathcal{G}_n$ and $(\mu_1) \subsetneq (\mu_2)$, then there is an element $e \in (\mu_2) \setminus (\mu_1)$ and a neutrosophic set $\mu_3 \in \mathcal{G}_n$ with

$$(\mu_3) = (\mu_1) \cup \{e\}, \quad T_{\mu_3}(f) \geq \tau \quad \forall f \in (\mu_3).$$

Typically one takes

$$\mathcal{G}_n = \{ \mu : (\mu) \text{ forms an initial segment of a reliable path in a network and } T(e) \geq \tau \}.$$

Example 4.9 (Greedoid of Path Construction). Let $E = \{e_1, e_2, e_3\}$ be edges of a simple directed tree from root r toward leaves, with reliabilities

$$T(e_1) = 0.9, \quad T(e_2) = 0.8, \quad T(e_3) = 0.7, \quad I(e) = 0.1, \quad F(e) = 1 - T(e) - 0.1,$$

and $\tau = 0.75$. Feasible supports are initial edge-sets whose all edges are reliable:

$$\mathcal{G}_n = \{ \emptyset, \{e_1\}, \{e_2\}, \{e_1, e_2\} \}.$$

- Deletion (NG1): Removing the last edge from any support leaves a shorter initial path still in $\mathcal{G}_n$.
- Expansion (NG2): From $\{e_1\} \subset \{e_1, e_2\}$, we can add e_2 (since $T(e_2) = 0.8 \geq 0.75$) to restore a feasible path.

Thus $(E, \mathcal{G}_n)$ is a neutrosophic greedoid capturing the process of building a reliable route one edge at a time.

4.3 Some Properties of Neutrosophic Matroids, Antimatroids, and Greedoids

Next, we explore how a Neutrosophic Matroid induces a classical matroid and a fuzzy matroid.

Theorem 4.10. Let (E, Ψ) be a neutrosophic matroid. Define

$$\mathcal{I} = \{ (\mu) : \mu \in \Psi \}.$$

Then $(E, \mathcal{I})$ satisfies all three axioms of a classical matroid, with $\mathcal{I}$ as its family of independent sets.

Proof. We check each matroid axiom in turn, adding intermediate steps for clarity.

Axiom (I1) Non-emptiness.

1. Existence of at least one independent set. By assumption $\Psi \neq \emptyset$, so choose any $\mu \in \Psi$. Then $(\mu) \in \mathcal{I}$.

2. Empty set is independent. Define μ_0 by

$$T_{\mu_0}(e) = 0, \quad I_{\mu_0}(e) = 1, \quad F_{\mu_0}(e) = 1 \quad \forall e \in E.$$

Clearly $\mu_0 \leq \mu$, so by (NM1) $\mu_0 \in \Psi$. Hence $(\mu_0) = \emptyset \in \mathcal{I}$.

Axiom (I2) Hereditary Property.

1. Let $A \in \mathcal{I}$ and $B \subseteq A$. Then there exists $\mu \in \Psi$ with $(\mu) = A$.
2. Define a neutrosophic set ν by

$$\nu(e) = \begin{cases} (T_\mu(e), I_\mu(e), F_\mu(e)), & e \in B, \\ (0, 1, 1), & e \notin B. \end{cases}$$

3. For each $e \in E$:

$$T_\nu(e) \leq T_\mu(e), \quad I_\nu(e) \geq I_\mu(e), \quad F_\nu(e) \geq F_\mu(e),$$

so $\nu \leq \mu$. By (NM1), $\nu \in \Psi$.

4. Therefore $(\nu) = B \in \mathcal{I}$, proving hereditary closure.

Axiom (I3) Exchange Property.

1. Let $A, B \in \mathcal{I}$ with $|A| < |B|$. Then pick $\mu, \nu \in \Psi$ so that $(\mu) = A$ and $(\nu) = B$.
2. By (NM2), there exists $\omega \in \Psi$ such that

$$\mu < \omega \leq \mu \vee \nu, \quad m_T(\omega) \geq \min\{m_T(\mu), m_T(\nu)\}.$$

3. Since $\mu < \omega$, there is some $e \in E$ with $T_\omega(e) > T_\mu(e)$. But $T_\mu(e) = 0$ whenever $e \notin A$, so necessarily $e \in B \setminus A$.
4. Hence

$$(\omega) = A \cup \{e\},$$

which has size $|A| + 1$. In particular, $(\omega) \in \mathcal{I}$ and $(\omega) \setminus A = \{e\} \subseteq B \setminus A$.

Since all three axioms are verified with these explicit intermediate steps, $(E, \mathcal{I})$ is indeed a classical matroid. $\square$

Theorem 4.11. Let (E, Ψ) be a Neutrosophic Matroid, and define the map

$$\Phi: \Psi \longrightarrow F(E), \quad \Phi(\mu) = \tau_\mu,$$

where τ_μ is the fuzzy set on E given by

$$\tau_\mu(e) = T_\mu(e), \quad e \in E.$$

Let $I = \Phi(\Psi) \subseteq F(E)$. Then (E, I) is a fuzzy matroid.

Proof. We check the two fuzzy matroid axioms for I .

Fuzzy Hereditary Property. Suppose $\tau_\mu \in I$, and let $\tau_\nu \leq \tau_\mu$ in the fuzzy-set order, i.e. $\tau_\nu(e) \leq \tau_\mu(e)$ for all $e \in E$. We must exhibit $\nu \in \Psi$ such that $\Phi(\nu) = \tau_\nu$. Define ν by

$$\begin{cases} T_\nu(e) = \tau_\nu(e), \\ I_\nu(e) = I_\mu(e), \\ F_\nu(e) = F_\mu(e), \end{cases} \quad e \in E.$$

Since $\tau_\nu(e) = T_\nu(e) \leq T_\mu(e)$ and $I_\nu(e) = I_\mu(e) \geq I_\mu(e)$, $F_\nu(e) = F_\mu(e) \geq F_\mu(e)$, we have $\nu \leq \mu$. By (NM1), $\nu \in \Psi$, so $\tau_\nu = \Phi(\nu) \in I$.

Fuzzy Exchange Property. Let $\tau_{\mu_1}, \tau_{\mu_2} \in I$ satisfy $|\text{supp}(\tau_{\mu_1})| < |\text{supp}(\tau_{\mu_2})|$. Then $\mu_1, \mu_2 \in \Psi$ with $\Phi(\mu_i) = \tau_{\mu_i}$. By (NM2), there is $\omega \in \Psi$ such that

$$\mu_1 < \omega \leq \mu_1 \vee \mu_2, \quad m_T(\omega) \geq \min\{m_T(\mu_1), m_T(\mu_2)\}.$$

Define $\tau_\omega = \Phi(\omega)$. Clearly $\tau_\omega \in I$. Since $\mu_1 < \omega$, some $e \in E$ has $T_\omega(e) > T_{\mu_1}(e)$. In particular, $e \in \text{supp}(\tau_{\mu_2}) \setminus \text{supp}(\tau_{\mu_1})$. Hence

$$\text{supp}(\tau_\omega) = \text{supp}(\tau_{\mu_1}) \cup \{e\},$$

verifying the fuzzy exchange condition. Therefore, (E, I) is a fuzzy matroid. $\square$

The next two theorems show how a Neutrosophic Antimatroid induces a classical antimatroid and a fuzzy antimatroid.

Theorem 4.12. Let $A = (E, \mathcal{F}_n)$ be a Neutrosophic Antimatroid. Define

$$\mathcal{F} = \{\text{supp}(\mu) : \mu \in \mathcal{F}_n\}.$$

Then $(E, \mathcal{F})$ is a classical antimatroid, with $\mathcal{F}$ as the family of feasible sets.

Proof. We check the three antimatroid axioms for $\mathcal{F}$.

1. **(i) Hereditary Property:** Let $F \in \mathcal{F}$, and let $F' \subseteq F$. Then there is $\mu \in \mathcal{F}_n$ with $\text{supp}(\mu) = F$. Define ν by

$$\begin{cases} T_\nu(e) = T_\mu(e), \quad I_\nu(e) = I_\mu(e), \quad F_\nu(e) = F_\mu(e), & e \in F', \\ T_\nu(e) = 0, \quad I_\nu(e) = 1, \quad F_\nu(e) = 1, & e \in E \setminus F'. \end{cases}$$

Since $\nu \leq \mu$, the Neutrosophic Deletion Property implies $\nu \in \mathcal{F}_n$. Hence $\text{supp}(\nu) = F' \in \mathcal{F}$.

2. **(ii) Accessibility Property:** Let $F \in \mathcal{F}$ with $F \neq \emptyset$. Choose $\mu \in \mathcal{F}_n$ such that $\text{supp}(\mu) = F$. By the Neutrosophic Deletion Property, there exists $x \in F$ for which $\mu_x = \mu \setminus \{x\} \in \mathcal{F}_n$. Hence $\text{supp}(\mu_x) = F \setminus \{x\} \in \mathcal{F}$.
3. **(iii) Union Closure:** Take $F_1, F_2 \in \mathcal{F}$ with $F_i = \text{supp}(\mu_i)$ and $\mu_i \in \mathcal{F}_n$. Let $\mu = \mu_1 \vee \mu_2$. By the Neutrosophic Union Closure Property, $\mu \in \mathcal{F}_n$. Then

$$\text{supp}(\mu) = \text{supp}(\mu_1) \cup \text{supp}(\mu_2) = F_1 \cup F_2,$$

so $F_1 \cup F_2 \in \mathcal{F}$.

Thus $(E, \mathcal{F})$ is a classical antimatroid. $\square$

Theorem 4.13. Let $A = (E, \mathcal{F}_n)$ be a Neutrosophic Antimatroid, and define

$$\Phi: \mathcal{F}_n \longrightarrow F(E), \quad \Phi(\mu) = \tau_\mu, \quad \tau_\mu(e) = T_\mu(e).$$

Let $\mathcal{F}_f = \Phi(\mathcal{F}_n)$. Then $(E, \mathcal{F}_f)$ is a fuzzy antimatroid.

Proof. We verify the two fuzzy antimatroid axioms for $\mathcal{F}_f$.

Fuzzy Deletion Property. Take $\tau_\mu \in \mathcal{F}_f$ with $\text{supp}(\tau_\mu) \neq \emptyset$. Since $\mu \in \mathcal{F}_n$, by the Neutrosophic Deletion Property there is an element $x \in \text{supp}(\mu)$ with $\mu_x = \mu \setminus \{x\} \in \mathcal{F}_n$. Then $\tau_{\mu_x} = \Phi(\mu_x)$ satisfies $\text{supp}(\tau_{\mu_x}) = \text{supp}(\tau_\mu) \setminus \{x\}$, so $\tau_{\mu_x} \in \mathcal{F}_f$.

Fuzzy Union Closure Property. Let $\tau_{\mu_1}, \tau_{\mu_2} \in \mathcal{F}_f$. Then $\mu_1, \mu_2 \in \mathcal{F}_n$. By the Neutrosophic Union Closure Property, $\mu = \mu_1 \vee \mu_2 \in \mathcal{F}_n$. Hence $\tau_\mu = \Phi(\mu)$ lies in $\mathcal{F}_f$, and

$$\tau_\mu(e) = T_\mu(e) = \max\{T_{\mu_1}(e), T_{\mu_2}(e)\} = (\tau_{\mu_1} \vee \tau_{\mu_2})(e).$$

Therefore $\tau_{\mu_1} \vee \tau_{\mu_2} \in \mathcal{F}_f$, verifying the fuzzy union closure. $\square$

Next, we show how a Neutrosophic Greedoid generates a classical greedoid and a fuzzy greedoid.

Theorem 4.14. Let $G = (E, \mathcal{G}_n)$ be a Neutrosophic Greedoid. Define

$$\mathcal{G} = \{\text{supp}(\mu) : \mu \in \mathcal{G}_n\}.$$

Then $(E, \mathcal{G})$ is a classical greedoid, with $\mathcal{G}$ as its family of feasible sets.

Proof. We check the three greedoid axioms for $\mathcal{G}$.

1. **(i) Empty-set Feasible:** Since $\mathcal{G}_n \neq \emptyset$, pick $\mu \in \mathcal{G}_n$. If $\text{supp}(\mu) \neq \emptyset$, then by repeated application of the Neutrosophic Deletion Property we obtain a neutrosophic set whose support is empty. Hence $\emptyset \in \mathcal{G}$.
2. **(ii) Deletion Property:** Let $F \in \mathcal{G}$ with $F \neq \emptyset$. Then there is $\mu \in \mathcal{G}_n$ such that $\text{supp}(\mu) = F$. The Neutrosophic Deletion Property yields $x \in F$ with $\mu_x = \mu \setminus \{x\} \in \mathcal{G}_n$. Hence $\text{supp}(\mu_x) = F \setminus \{x\} \in \mathcal{G}$.
3. **(iii) Exchange Property:** Let $F, F' \in \mathcal{G}$ satisfy $|F| < |F'|$. Then choose $\mu, \mu' \in \mathcal{G}_n$ with $\text{supp}(\mu) = F$ and $\text{supp}(\mu') = F'$. Since $\text{supp}(\mu) \subseteq \text{supp}(\mu')$ and $\mu \leq \mu'$, by the Neutrosophic Expansion Property there exists $\omega \in \mathcal{G}_n$ such that

$$\mu \leq \omega \leq \mu \vee \mu', \quad m_T(\omega) \geq \min\{m_T(\mu), m_T(\mu')\}.$$

Because $\mu < \omega$, there is an element $e \in E$ with $T_\omega(e) > T_\mu(e)$. In particular, $e \in F' \setminus F$. Thus $\text{supp}(\omega) = F \cup \{e\} \in \mathcal{G}$.

Therefore, $(E, \mathcal{G})$ is a classical greedoid. $\square$

Theorem 4.15. Let $G = (E, \mathcal{G}_n)$ be a Neutrosophic Greedoid, and define

$$\Phi: \mathcal{G}_n \longrightarrow F(E), \quad \Phi(\mu) = \tau_\mu, \quad \tau_\mu(e) = T_\mu(e).$$

Let $\mathcal{G}_f = \Phi(\mathcal{G}_n)$. Then $(E, \mathcal{G}_f)$ is a fuzzy greedoid.

Proof. We verify the two fuzzy greedoid axioms for $\mathcal{G}_f$.

Deletion Property. Take $\tau_\mu \in \mathcal{G}_f$ with $\text{supp}(\tau_\mu) \neq \emptyset$. Since $\mu \in \mathcal{G}_n$, by the Neutrosophic Deletion Property there is $x \in \text{supp}(\mu)$ such that $\mu_x = \mu \setminus \{x\} \in \mathcal{G}_n$. Therefore $\tau_{\mu_x} = \Phi(\mu_x) \in \mathcal{G}_f$, and $\text{supp}(\tau_{\mu_x}) = \text{supp}(\tau_\mu) \setminus \{x\}$.

Expansion Property. Let $\tau_{\mu_1}, \tau_{\mu_2} \in \mathcal{G}_f$ with $\text{supp}(\tau_{\mu_1}) \subseteq \text{supp}(\tau_{\mu_2})$ and $\tau_{\mu_1} \leq \tau_{\mu_2}$. Then $\mu_1, \mu_2 \in \mathcal{G}_n$. By the Neutrosophic Expansion Property, there exists $\mu_3 \in \mathcal{G}_n$ such that

$$\mu_1 \leq \mu_3 \leq \mu_1 \vee \mu_2, \quad m_T(\mu_3) \geq \min\{m_T(\mu_1), m_T(\mu_2)\}.$$

Set $\tau_{\mu_3} = \Phi(\mu_3)$. Then $\tau_{\mu_3} \in \mathcal{G}_f$, and $\tau_{\mu_1} \leq \tau_{\mu_3} \leq \tau_{\mu_1} \vee \tau_{\mu_2}$.

Thus $(E, \mathcal{G}_f)$ is a fuzzy greedoid. □

5 Conclusion and Future Tasks

5.1 Summary of Results

This paper introduced formal mathematical definitions for Neutrosophic Matroids, Neutrosophic Antimatroids, and Neutrosophic Greedoids. These combinatorial structures generalize their classical counterparts by incorporating neutrosophic sets, thereby enabling more expressive modeling of uncertainty, indeterminacy, and partial truth. We also found that Neutrosophic Matroids, Antimatroids, and Greedoids can further generalize existing frameworks such as Fuzzy Matroids, Fuzzy Antimatroids, and Fuzzy Greedoids. Given that matroids are closely related to fundamental structures like spanning trees, future work may include investigating concrete real-world applications and constructive algorithms for Neutrosophic Matroids and their variants.

5.2 Future Work

Future research will focus on extending these frameworks within the context of Plithogenic Sets (cf. [67, 68, 69, 70, 71]). By leveraging the principles of Plithogenic theory—which accounts for complex uncertainty through the degree of contradiction among attribute values—one can further generalize matroids, antimatroids, and greedoids into Plithogenic Matroids, Plithogenic Antimatroids, and Plithogenic Greedoids, respectively. These enriched structures are expected to offer a more flexible and nuanced foundation for applications requiring advanced uncertainty modeling.

In addition, future directions include exploring potential extensions using the frameworks of Hyperstructures and SuperHyperstructures [72, 73, 74, 75, 76, 77, 78, 79], with the aim of enhancing the hierarchical and compositional expressiveness of these neutrosophic structures.

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials. To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

Competing interests

Author has declared that no competing interests exist.

Consent to Publish declaration

The author approved to Publish declarations.

References

- [1] James G Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.
- [2] Petr Hliněný and Geoff Whittle. Matroid tree-width. *European Journal of Combinatorics*, 27(7):1117–1128, 2006.
- [3] András Recski. *Matroid theory and its applications in electric network theory and in statics*, volume 6. Springer Science & Business Media, 2013.
- [4] Takaaki Fujita. Matroid, ideal, ultrafilter, tangle, and so on: Reconsideration of obstruction to linear decomposition. *International Journal of Mathematics Trends and Technology (IJMTT)*, 70:18–29, 2024.
- [5] Hua Mao and Sanyang Liu. On antimatroids of infinite character. *Mathematica Pannonica*, 23(2):257–266, 2012.
- [6] E Algaba, Jesús Mario Bilbao, Rene Van den Brink, and Andres Jimenez-Losada. Cooperative games on antimatroids. *Discrete Mathematics*, 282(1-3):1–15, 2004.
- [7] Anders Björner and Günter M Ziegler. Introduction to greedoids. *Matroid applications*, 40:284–357, 1992.
- [8] Bernhard Korte and László Lovász. Structural properties of greedoids. *Combinatorica*, 3(3):359–374, 1983.
- [9] Bernhard Korte and László Lovász. Polymatroid greedoids. *Journal of Combinatorial Theory, Series B*, 38(1):41–72, 1985.
- [10] Bernhard Korte, László Lovász, and Rainer Schrader. *Greedoids*, volume 4. Springer Science & Business Media, 2012.
- [11] Takaaki Fujita. The interplay between quasi-matroid and connectivity systems. *Asian Research Journal of Mathematics*, 21(1):115–129, 2025.
- [12] Yukihiro Kawahara. On matroids and orlik-solomon algebras. *Annals of Combinatorics*, 8:63–80, 2004.
- [13] Takaaki Fujita. Relation between ultra matroid and linear decomposition. *Italian Journal of Pure and Applied Mathematics*, page 18, 2024.
- [14] Takaaki Fujita. Semi-matroids on connectivity system and linear decomposition. *Asian Research Journal of Mathematics*, 21(4):1–13, 2025.
- [15] Takaaki Fujita. On ultrafilters in matroids: Structural properties and metric considerations. *Asian Research Journal of Mathematics*, 21(3):106–117, 2025.
- [16] Jim Geelen, Bert Gerards, and Geoff Whittle. Obstructions to branch-decomposition of matroids. *Journal of Combinatorial Theory, Series B*, 96(4):560–570, 2006.
- [17] Muhammad Akram and Bijan Davvaz. Strong intuitionistic fuzzy graphs. *Filomat*, 26(1):177–196, 2012.
- [18] Muhammad Akram, Sheng-Gang Li, and KP Shum. Antipodal bipolar fuzzy graphs. *Italian Journal of Pure and Applied Mathematics*, 31(56):425–438, 2013.
- [19] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.
- [20] Supriya Kumar De, Ranjit Biswas, and Akhil Ranjan Roy. Some operations on intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 114(3):477–484, 2000.
- [21] TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.
- [22] Tareq M Al-shami, Salem Saleh, Alaa M Abd El-latif, and Abdelwaheb Mhemdi. Novel categories of spaces in the frame of fuzzy soft topologies. *AIMS Press*, 9(3):6305–6320, 2024.
- [23] Eulalia Szmidt. *Distances and similarities in intuitionistic fuzzy sets*, volume 307. Springer, 2014.
- [24] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.
- [25] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.
- [26] Gia Nhu Nguyen, Le Hoang Son, Amira S Ashour, and Nilanjan Dey. A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. *International Journal of Machine Learning and Cybernetics*, 10:1–13, 2019.
- [27] Florentin Smarandache, Memet Şahin, Derya Bakkak, Vakkas Uluçay, and Abdullah Kargın. *Neutrosophic SuperHyperAlgebra and New Types of Topologies*. Infinite Study, 2023.
- [28] Florentin Smarandache. Neutrosophic physics: More problems, more solutions. 2010.
- [29] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.
- [30] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.
- [31] S Saleh. On intuitionistic fuzzy separation axioms. *Abhath Journal of Basic and Applied Sciences*, 1(1):37–56, 2022.

- [32] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.
- [33] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.
- [34] Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. *Journal of Intelligent & Fuzzy Systems*, 30(6):3675–3680, 2016.
- [35] Takaaki Fujita and Florentin Smarandache. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, 80:609–631, 2025.
- [36] Takaaki Fujita and Florentin Smarandache. Examples of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets in climate change and the proposal of several new concepts. *Climate Change Reports*, 2:1–18, 2025.
- [37] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.
- [38] Takaaki Fujita and Prem Kumar Singh. Hyperfuzzy graph and hyperfuzzy hypergraph. *Journal of Neutrosophic and Fuzzy Systems (JNFS)*, 10(01):01–13, 2025.
- [39] Florentin Smarandache. *Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics*. Infinite Study, 2017.
- [40] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.
- [41] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.
- [42] Fu-Gui Shi. A new approach to the fuzzification of matroids. *Fuzzy Sets and Systems*, 160(5):696–705, 2009.
- [43] Deyin Wu and Yonghong Li. The induced basis axioms for a closed gv fuzzy matroid. *Journal of Intelligent & Fuzzy Systems*, 40(1):1037–1049, 2021.
- [44] Hui-Min Xiao, Mei-Qi Wang, Yan-Li Cao, and Yu-Jie Guo. Research on hesitant fuzzy clustering method based on fuzzy matroids. *Journal of Intelligent & Fuzzy Systems*, 40(5):9031–9039, 2021.
- [45] Deyin WU. On calculations of fuzzy ranks in closed fuzzy matroids. *Journal of Southwest China Normal University (Natural Science Edition)*, 46(12):5–12, 2021.
- [46] OK Shabna and K Sameena. Graphic fuzzy matroids. *South East Asian J. of Mathematics and Mathematical Sciences*, 17(1):223–232, 2021.
- [47] K Sameena et al. Fuzzy matroids from fuzzy vector spaces. *South East Asian Journal of Mathematics and Mathematical Sciences*, 17(03):381–390, 2021.
- [48] Muhammad Asif, Doha A Kattan, Dragan Pamučar, and Ghous Ali. q-rung orthopair fuzzy matroids with application to human trafficking. *Discrete Dynamics in Nature and Society*, 2021(1):8261118, 2021.
- [49] Fu-Gui Shi. (l, m)-fuzzy matroids. *Fuzzy sets and systems*, 160(16):2387–2400, 2009.
- [50] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.
- [51] Jack Edmonds. Matroid intersection. In *Annals of discrete Mathematics*, volume 4, pages 39–49. Elsevier, 1979.
- [52] Robert E Bixby and William H Cunningham. Matroid optimization and algorithms. 1990.
- [53] Jack Edmonds. Matroid partition. *50 Years of Integer Programming 1958-2008: From the Early Years to the State-of-the-Art*, pages 199–217, 2010.
- [54] Alexander Heaton and José Alejandro Samper. Dual matroid polytopes and internal activity of independence complexes. *arXiv preprint arXiv:2005.04252*, 2020.
- [55] Paul D. Seymour. Recognizing graphic matroids. *Combinatorica*, 1:75–78, 1981.
- [56] Neil White. *Matroid applications*. Number 40. Cambridge University Press, 1992.
- [57] Fedor V Fomin, Sang-il Oum, and Dimitrios M Thilikos. Rank-width and tree-width of h-minor-free graphs. *European Journal of Combinatorics*, 31(7):1617–1628, 2010.
- [58] Vašek Chvátal. Antimatroids, betweenness, convexity. *Research Trends in Combinatorial Optimization: Bonn 2008*, pages 57–64, 2009.
- [59] Fu-Ning Lin, Guang-Ji Yu, Guang-Ming Xue, and Jiang-Feng Han. L-fuzzifying antimatroids: A fuzzy approach to the generalization of shelling precedence structures. *Journal of Intelligent & Fuzzy Systems*, 39(3):4183–4196, 2020.
- [60] Steven J Tedford et al. Rank functions of fuzzy greedoids. *Open Journal of Discrete Mathematics*, 5(04):65, 2015.

- [61] Zhen-Yu Xiu and Fu-Gui Shi. Categories of (i, i) -fuzzy greedoids. *Journal of Function Spaces*, 2015(1):543247, 2015.
- [62] Muhammad Akram and Musavarah Sarwar. Novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. *viXra*, 2017.
- [63] Gulfam Shahzadi, Muhammad Akram, Arsham Borumand Saeid, et al. An application of single-valued neutrosophic sets in medical diagnosis. *Neutrosophic sets and systems*, 18:80–88, 2017.
- [64] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.
- [65] Jia Syuen Chai, Ganeshsree Selvachandran, Florentin Smarandache, Vassilis C Gerogiannis, Le Hoang Son, Quang-Thinh Bui, and Bay Vo. New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems. *Complex & Intelligent Systems*, 7:703–723, 2021.
- [66] Faruk Karaaslan and Fatih Hunu. Type-2 single-valued neutrosophic sets and their applications in multi-criteria group decision making based on topsis method. *Journal of Ambient Intelligence and Humanized Computing*, 11:4113 – 4132, 2020.
- [67] Muhammad Azeem, Humera Rashid, Muhammad Kamran Jamil, Selma Gütmen, and Erfan Babaei Tirkolaei. Plithogenic fuzzy graph: A study of fundamental properties and potential applications. *Journal of Dynamics and Games*, pages 0–0, 2024.
- [68] S Gomathy, D Nagarajan, S Broumi, and M Lathamaheswari. *Plithogenic sets and their application in decision making*. Infinite Study, 2020.
- [69] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.
- [70] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.
- [71] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.
- [72] Ajoy Kanti Das, Rajat Das, Suman Das, Bijoy Krishna Debnath, Carlos Granados, Bimal Shil, and Rakhal Das. A comprehensive study of neutrosophic superhyper bci-semigroups and their algebraic significance. *Transactions on Fuzzy Sets and Systems*, 8(2):80, 2025.
- [73] Prabakaran Raghavendran and Tharmalingam Gunasekar. Optimizing organ transplantation success using neutrosophic superhyperstructure and artificial intelligence. *Volume IV*, page 117.
- [74] Adel Al-Odhari. A brief comparative study on hyperstructure, super hyperstructure, and n-super superhyperstructure. *Neutrosophic Knowledge*, 6:38–49, 2025.
- [75] Takaaki Fujita. Antihyperstructure, neurohyperstructure, and superhyperstructure. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 311, 2025.
- [76] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.
- [77] Takaaki Fujita. Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 416, 2025.
- [78] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.
- [79] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.