CERTAIN TYPES OF K^h-BIRECURRENT FINSLER SPACE(I)

by

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Abstract

H.D. Pande and B. Singh [21] discussed the recurrency in an affinely connected space. P.K. Dwivedi [6] worked out the role of P^*-reducible space in affinely connected space. A.A.A. Muhib [20] obtained some results when R^h-generalized trirecurrent and R^h −special generalized trirecurrent spaces are affinely connected spaces. A.A. M. Saleem [24] obtained some results when the C^h-recurrentin C^h-generalized birecurrent and C^h-special generalized birecurrent are affinely connected spaces.

A Landsberg space of dimension 2 was first considered by G. Landsberg [14] from a standpoint of variation. As to such spaces of many dimensions, É. Cartan [5] introduced it as one of particular cases and further L. Berwald ([2],[3]) showed that the space was characterized by $P^i_{jkh} = 0$, where P^i_{jkh} is the (hv)-curvature tensor. H. Yasuda [26] gave other characterizations of a Landsberg space and contributed a little to the theory of Landsberg spaces. R. Verma [25] obtained the condition of a P-reducible R^h-recurrent space be a necessarily a Landsberg space. A.A.M. Saleem [24] obtained some results when the C^h-recurrentin C^h-generalized birecurrent and C^h-special generalized birecurrent spaces are Landsberg spaces. P.K. Dwivedi [6] worked out the role of P^*-reducible space in Landsberg space .

In this paper we use the property of k^h-BR-F_n in affinely connected space and Landsberg space. We have obtained different

theorems for some tensors to be satisfying the conditions of the above spaces and we have obtained various identities in such spaces.

Keywords: K^h-Birecurrent Space , K^h-Birecurrent Affinely Connected Space and K^h-Landsberg space.

1. Introduction

Let us consider an n-dimensional Finsler space F_n equipped with a metric function F(x^i, y^i) * satisfying the requestic conditions of a Finslerian metric [21] . The relations between the metric function F and the corresponding metric tensor g_{ij} ** are given by

(1.1) a) $g_{ij} = \frac{1}{2}\, \dot{\partial}_i\, \dot{\partial}_j\, F^2$ *** and b) $g_{ij}\, y^i\, y^j = F^2$.

The totality of all such vectors associated with point P of F_n is known as *dual tangent space* at P and denoted by $\bar{T}_n(p)$, the metric function of the dual tangent space is Hamiltonian function H(x^i, y_i) satisfying the three requisite conditions required for a Finsler space.

Analogous to the metric tensor $g_{ij}(x, y)$, we define a tensor $g^{ik}(x^k, y_k)$ as follows :

(1.2) $$g^{ik}(x^k, y_k) = \frac{1}{2}\; \bar{\partial}_i\, \bar{\partial}_j\; H^2(x^k, y_k)\ ,$$

where $\bar{\partial}_i$ denoted the partial differentiation w.r.t. covariant vector y_i . The quantities $g^{ik}(x^k, y_k)$constitute the components of a contravariant tensor of second order. The quantities g_{ij} *and* g^{ij} which are components of the metric tensor and associate metric tensor are connected by

(1.3) $$g_{ij}\;\; g^{ij} = \delta_i^k = \begin{cases} 1 & ,if \quad i = k\ , \\ 0 & ,if \quad i \neq k\ . \end{cases}$$

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* Unless stated otherwise . Hence forth all geometric objects are assumed to be functions of line–elements.

** Indices i , j , k , … assumed positive integer values from 1 to n .

*** $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

From the metric tensor we construct a new tensor C_{ijk} by diff.(1.1a) partially w.r.t. y^k , we get

(1.4) $$C_{ijk} \coloneqq \frac{1}{2}\, \dot{\partial}_i\, g_{jk} = \frac{1}{4}\, \dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2 \ ,$$

where C_{ijk} is known as *(h) hv- torsion tensor* [14] ,it is positively homogenous of degree -1 in y^i and symmetric in all its indices . The *(v) hv- torsion tensor* C^i_{jk} *is the associate tensor of the tensor* C_{ijk} is defined by

(1.5) a) $C^h_{ik} \coloneqq g^{hj}\ C_{ijk}$ and b) $C^i_{ji} = C_j$

which is positively homogenous of degree -1 in y^i and symmetric in its lower indices .

É. Cartan deduced ([4],[5])

(1.6) $$X^i_{|k} \coloneqq \partial_k X^i - \left(\dot{\partial}_r X^i\right) G^r_k + X^r\ \Gamma^{*i}_{rk}$$

for an arbitrary vector field X^i , where Γ^{*i}_{rk} is Cartan's connection parameter and the function G^r_k is positively homogenous of degree one in y^i.

The metric tensor g_{ij} and its associate tensor g^{ij} are covariant constant w.r.t. above process , i.e.

(1.7) a) $g_{ij|k} = 0$ and b) $g^{ij}_{|k} = 0$.

Also , the vector y^i vanish under h-covariant differentiation , i.e.

(1.8) $$y^i_{|k} = 0 .$$

The v (hv)- torsion tensor is defined by

(1.9) $$P^r_{jk} \coloneqq \left(\dot{\partial}_j\ \Gamma^{*i}_{rk}\right) y^h \coloneqq \Gamma^{*r}_{jkh}\ y^h \quad .$$

The tensor K^i_{rkh} is called *Cartan's fourth curvature tensor* and defined as

(1.9a) $$K^i_{rkh} \coloneqq \partial_h\ \Gamma^{*i}_{kr} + \left(\dot{\partial}_l\ \Gamma^{*i}_{rh}\right) G^l_k + \Gamma^{*i}_{mh}\ \Gamma^{*m}_{kr} - \mathrm{h/k}^*$$

which is skew-symmetric in its last two lower indices k and h , i.e.

(1.9b) $$K^{i}_{jkh} = - K^{i}_{jhk}$$

and satisfies the following identity known as Bianchi identity

*- h / k means the subtraction from the former term by interchanging the indices h and k.

(1.9c) $$K^{r}_{ihk|j} + K^{r}_{ijh|k} + K^{r}_{ikj|h} + \left(\dot{\partial}_s \, \Gamma^{*r}_{ij}\right) K^{s}_{thk} \; y^t + \left(\dot{\partial}_s \, \Gamma^{*r}_{ik}\right) K^{s}_{tjh} \; y^t + \left(\dot{\partial}_s \, \Gamma^{*r}_{ih}\right) K^{s}_{tkj} \; y^t = 0 .$$

The associate tensor K_{ijkh} of the curvature tensor K^{i}_{jkh} is given by

(1.10) $$K_{ijkh} := g_{rj} \; K^{r}_{jkh} \; .$$

The Ricci tensor K_{jk} of the curvature tensor K^{i}_{jkh} is given by

(1.11) $$K^{i}_{jki} = K_{jk} \; .$$

The curvature tensor K^{i}_{jkh} satisfies the following relations too

(1.12) $a)$ $$K^{i}_{jkh} \; y^j = H^{i}_{kh}$$

and b) $$H^{i}_{mkh} - K^{i}_{mkh} = P^{i}_{mk|h} + P^{r}_{mk} P^{i}_{rh} - \text{h/k} ,$$

where the quantities H^{i}_{mkh} and H^{i}_{kh} form Berwald curvature tensor and h(v)- torsion tensor respectively.

The tensor R^{i}_{jkh} called *h- curvature tensor (Cartan's third curvature tensor)* and defined as

(1.13a) $$R^{i}_{jkh} := \partial_h \, \Gamma^{*i}_{jk} + \left(\dot{\partial}_l \, \Gamma^{*i}_{jh}\right) G^{l}_{k} + G^{i}_{jm}(\partial_h \, G^{m}_{h} - G^{m}_{hl} \, G^{l}_{k}) + \Gamma^{*i}_{mh} \, \Gamma^{*m}_{jk} - \text{h / k} ,$$

this tensor is positively homogenous of degree -1 in y^i and skew-symmetric in its last two lower indices k and h , i.e.

(1.13b) $$R^{i}_{jkh} = - R^{i}_{jhk} \; .$$

The associate tensor R_{ijkh} of the curvature tensor R^{i}_{jkh} is given by

(1.14) $$R_{ijkh} := g_{rj}\ R^r_{ikh} \quad .$$

The Ricci tensor R_{jk} of the curvature tensor R^i_{jkh} and the tensor R^r_h are given by

(1.15) a) $R^i_{jki} = R_{jk}$ and b) $R^r_{ikh}\, g^{ik} = R^r_h$.

The curvature tensor R^i_{jkh} satisfies the following identity known as Bianchi identity

(1.16) $$R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + y^m \left(R^s_{mkh}\, P^r_{ijs} + R^s_{mjk}\, P^r_{ihs} + R^s_{mhj}\, P^r_{iks}\right) = 0 ,$$

where P^i_{jkh} is called *hv-curvature tensor (Cartan's second curvature tensor)* and positively homogenous of degree zero in y^i and satisfies the relations

(1.17) a) $P^i_{jkh}\, y^j = \Gamma^{*i}_{jkh}\, y^j = P^i_{kh} = C^i_{kh|r}\ y^r$

and b) $P^i_{jkh}\ y^k = P^i_{jkh}\ y^h = 0$,

where P^i_{jk} is called *v (hv) - torsion tensor* .

Berwald curvature tensor H^i_{rkh} and the h(v) –torsion tensor H^i_{kh} are connected by

(1.18) a) $H^i_{rkh}\ y^r = H^i_{kh}$

and b) $H^i_{rkh} = \dot{\partial}_r\, H^i_{kh}$.

Berwald constructed initially the curvature tensor H^i_{jkh} and the torsion tensor H^i_{kh} by means of the tensor H^i_k called deviation tensor ([15],[18]) , according to

(1.19) a) $H^i_{jkh} = \frac{1}{3}\ \dot{\partial}_j\ (\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)$

and b) $H^i_{kh} = \frac{1}{3}(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)$,

where

c) $H_h^i := 2\,\partial_h G^i - \partial_s G_h^i\, y^s + 2\, G^s\, G_{hs}^i - G_s^i\, G_h^s$.

The deviation tensor H_k^i is positively homogenous of degree two in y^i and satisfies

(1.20a) $a)$ $H_k^i = H_{hk}^i y^h$.

The tensors H_{kh}^i and H_k^i satisfy

(1.20b) b) $y_i\, H_{kh}^i = 0$.

In view of Euler's theorem on homogenous functions we have the following relation

(1.21) $H_{jk}^i y^i = H_k^i = -H_{kj}^i y^j$.

The contraction of the indices i and h in (1.19a) , (1.19b)and (1.19c) yields the following :

(1.22) a) $H_{jk} = H_{jki}^i$, b) $H_k = H_{ki}^i$ and c) $H = \frac{1}{n-1} H_i^i$,

where H_{jk} and H are called *h- Ricci tensor*[22] and *curvature scalar* respectively.

By using (1.22a) , (1.22b) and (1.22c) these contracted tensors are connected by

(1.23) a) $H_{jk} = \partial_j H_k$, b) $H_{jk}\, y^j = H_k$

and c) $H_k\, y^k = (n-1)\, H$.

The tensor $H_{jh.k}$ defined by

(1.24) $H_{jh.k} := g_{\,ih}\ H_{jk}^i$.

2. K^h-Birecurrent Affinely Connected Space

Let us consider an K^h–birecurrent space which is characterized by

(2.1) $K_{jkh|m|\ell}^{\,i} = a_{\,\ell m} K_{jkh}^{\,i}$, $K_{jkh}^{\,i} \neq 0$, where $a_{\ell m}$ is a non–zero covariant tensor field of second order will be called *h-birecurrent tensor*. We shall denote such space and tensor briefly by k^h-*BR*- F_n and *h-BR* respectively .

Transvecting (2.1) by g_{ip} and using (1.6a) and (1.10) , we get

(2.2) $$K_{jpkh|m|l} = a_{lm} K_{jpkh} .$$

Transvecting (2.1) by y^j and using (1.7) and (1.12a) , we get

(2.3) $$H^i_{kh|m|l} = a_{lm} H^i_{kh} .$$

The associate tensor K_{ijkh} of Cartan's fourth curvature tensor $K^{\ i}_{jkh}$ is satisfying the identity [23]

(2.4) $$K_{ijhk} + K_{ikjh} + K_{ihkj}$$
$$= 2 \left(C_{ijs} K^s_{rhk} + C_{iks} K^s_{rjh} + C_{ihs} K^s_{rkj} \right) y^r .$$

(2.5) $$K_{ijkh} + K_{ikjh} + K_{ihkj} = 2(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj}) .$$

A Finsler space whose connection parameter G^i_{jk} is independent of y^i is called*an affinely connected space (Berwald space)*. Thus, an affinely connected space is characterized by any one of the following equivalent conditions

(2.6) a) $G^{\ i}_{jkh} = 0$

and

b) $C_{ijk|h} = 0$,

the connection parameters Γ^{*i}_{kh} of Cartan and $G^{\ i}_{kh}$ of Berwald coincides in affinely connected Finsler space and they are independent of directional arguments [23] ,i.e.

(2.7) a) $G^i_{jkh} = \dot{\partial}_j G^i_{kh} = 0$

and

b) $\dot{\partial}_j \Gamma^{*i}_{kh} = 0$.

Definition 2.1. If the K^h-birecurrent space is affinely connected space we called it K^h*-birecurrent affinely connected space* and denoted briefly by K^h*-BR-affinely connected space*.

Suppose the K^h-BR- F_n is affinely connected space and if $\dot{\partial}_j a_{\ell m} = 0$. By using these conditions in equ.{(3.2),[1]} Berwald curvature tensor H^i_{jkh} is *h-BR*, $i.e.$

(2.8) $$H^i_{jkh|m|\ell} = a_{\ell m} H^i_{jkh} .$$

Thus, we conclude

Theorem 2.1. *In K^h-BR- affinely connected space, if the directional derivative of covariant tensor field of second order vanish, then Berwald curvature tensor H^i_{jkh} is h-BR.*

Suppose the K^h-*BR*-F_n is affinely connected space and if $\dot{\partial}_j a_{\ell m} = 0$. By using these conditions in equ.{(3.7),[1]} the associate tensor H_{jpkh} of Berwald curvature tensor H^i_{jkh} is *h-BR*, $i.e.$

(2.9) $$H_{jpkh|m|\ell} = a_{\ell m} H_{jpkh} .$$

Thus, we conclude

Theorem 2.2. *In K^h-BR-affinely connected space, if the directional derivative of covariant tensor field of second order vanish , then the* associate tensor H_{jpkh} of *Berwald curvature tensor H^i_{jkh} is h-BR .*

Suppose the K^h-BR-F_n is affinely connected space and if $\dot{\partial}_j a_{\ell m} = 0$. By using these conditions in equ. { (3.10),[1] } the h-Ricci tensor H_{jk} is h-BR , $i.e.$

(2.10) $$H_{jk|m|\ell} = a_{\ell m} H_{jk} .$$

Thus, we conclude

Theorem 2.3. *In K^h-BR-affinely connected space, if the directional derivative of covariant tensor field of second order vanish , then the h-Ricci tensor H_{jk} is h-BR .*

Suppose the K^h-*BR*-F_n is affinely connected space and if $\dot{\partial}_j a_{\ell m} = 0$. By using these conditions in equ.{ (3.11),[1] } the tensor $(H_{hk} - H_{kh})$, is *h-BR*, i.e.

(2.11) $$(H_{hk} - H_{kh})_{|m|\ell} = a_{\ell m}(H_{hk} - H_{kh}) .$$

Thus, we conclude

Theorem 2.4. *In K^h-BR-affinely connected space, if the directional derivative of covariant tensor field of second order vanish , then the tensor $(H_{hk} - H_{kh})$, is h-BR .*

Transvecting equ.(2.8) by y^j and using (1.7) and (1.18a),we get

(2.12) $$H^i_{kh|m|\ell} = a_{\ell m} H^i_{kh} .$$

Transvecting equ.(2.12) by y^k and using (1.7) and (1.20a), we get

(2.13) $$H^i_{h|m|\ell} = a_{\ell m} H^i_h .$$

Contracting the indices i and h in (2.12) and using (1.22b),we get

(2.14) $$H_{k|m|\ell} = a_{\ell m} H_k .$$

Contracting the indices i and h in equ.(2.13) and using (1.22c),we get

(2.15) $$H_{|m|\ell} = a_{\ell m} H .$$

Transvecting (2.12) by g_{ip} and using(1.6a) and (1.24), we get

(2.16) $$H_{kp.h|m|\ell} = a_{\ell m} H_{kp.h} .$$

Thus, we conclude

Theorem 2.5. *In K^h-BR-affinely connected space, if the directional derivative of covariant tensor field of second order vanish, then the h(v)-torsion tensor H^i_{kh} ,the deviation tensor H^i_h, the vector H_k , the scalar H and the tensor $H_{kp.h}$ are all h-BR.*

Using (1.8) in { (3.16),[1] },we get

$$R^i_{jkh|m|l} = a_{lm} R^i_{jkh} + C^i_{jr|m|l} H^r_{kh} + C^i_{jr|m} H^r_{kh|l} + C^i_{jr|l} H^r_{kh|m} .$$

Suppose the K^h-BR-F_n is affinely connected space, then the above equ. can be written as

(2.17) $$R^i_{jkh|m|\ell} = a_{\ell m} R^i_{jkh} .$$

Transvecting (2.17) by g_{ip} and using (1.6a) and (1.14), we get

(2.18) $$R_{jpkh|m|\ell} = a_{\ell \mathrm{m}} R_{jpkh} .$$

Contracting the indices i and h in (2.17) and using (1.15a),we get

(2.19) $$R_{jk|m|\ell} = a_{\ell \mathrm{m}} R_{jk} .$$

Again, transvecting (2.17) by g^{jk} and using(1.6b) and (1.15b), we get

(2.20) $$R^i_{h|m|\ell} = a_{\ell \mathrm{m}}\, R^i_h .$$

Thus, we conclude

Theorem 2.6. *Cartan's third curvature tensor R^i_{jkh}, its associate tensor R_{jpkh} ,the Ricci tensor R_{jk} and the tensor R^i_h are all h-BR either the space is K^h-BR-F_n or K^h-BR- affinely connected space.*

Suppose the K^h-BR-F_n is affinely connected space. In view of (2.2b), the covariant derivative of the identity (1.9c) with respect to x^ℓ in the sense of Cartan and using{ (2.2),[1] }equ., gives

(2.21) $$a_{\ell m}\, K^i_{jkh} + a_{\ell h}\, K^i_{jmk} + a_{\ell k}\, K^i_{jhm} = 0 .$$

Thus, we conclude

Theorem 2.7. *In K^h-BR-affinely connected space, the identity (2.21) holds good .*

Contracting the indices i and m in (2.21) and using (1.9b) and (1.11), we get

(2.22) $$a_{\ell i}\, K^i_{jkh} - a_{\ell h}\, K_{jk} + a_{\ell k}\, K_{jh} = 0$$

which can be written as

(2.23) $$K^i_{jkh} = \frac{1}{a_{\ell i}}\left(a_{\ell h}\, K_{jk} - a_{\ell k}\, K_{jh}\right) .$$

Thus, we conclude

Theorem 2.8. *In K^h-BR-affinely connected space, Cartan's fourth curvature tensor K^i_{jkh} defined by the formula (2.23) .*

Suppose the K^h-BR-F_n is affinely connected space. In view of equ.(1.8) and (2.1b) the identity (1.12b) can be written as

(2.24) $$H^i_{jkh} = K^i_{jkh} \,.$$

Putting (2.24) in (2.21), we get

(2.25) $$a_{\ell m}\, H^i_{jkh} + a_{\ell h}\, H^i_{jmk} + a_{\ell k}\, H^i_{jhm} = 0 \,.$$

Transvecting of (2.25) by y^j and using (1.7) and (1.18a), we get

(2.26) $$a_{\ell m}\, H^i_{kh} + a_{\ell h}\, H^i_{mk} + a_{\ell k} H^i_{hm} = 0 \,.$$

Thus, we conclude

Theorem 2.9. *In K^h-BR-affinely connected space, the identities (2.24), (2.25) and (2.26) all hold good .*

Suppose the K^h-BR-F_n is affinely connected space. In view of equ.(1.17a) and equ. (2.1b) the Bianchi identity (1.16) for Cartan's third curvature tensor R^i_{jkh} can be written as

(2.27) $$R^r_{ijh|k} + R^r_{ikj|h} + R^r_{ihk|j} = 0 \,.$$

Diff. (2.27) covariantly w.r.t. x^ℓ in the sense of Cartan and using (2.17), we get

(2.28) $$a_{\ell k}\, R^r_{ijh} + a_{\ell h}\, R^r_{ikj} + a_{\ell j}\, R^r_{ihk} = 0 \,.$$

Contracting the indices r and h in (2.28) and using (1.13b) and (1.15a), we get

(2.29) $$a_{\ell r}\, R^r_{ikj} + a_{\ell k}\, R_{ij} - a_{\ell j}\, R_{ik} = 0$$

which can be written as

(2.30) $$R^r_{ikj} = \frac{1}{a_{\ell r}}\left(a_{\ell j} R_{ik} - a_{\ell k} R_{ij}\right) .$$

Thus, we conclude

Theorem 2.10. *In K^h-BR- affinely connected space, Cartan's third curvature tensor R^i_{jkh} is defined by the formula (3.30).*

3. K^h-Birecurrent Landsberg Space

Cartan's connection parameter Γ^{*i}_{jk} coincide with Berwald's connection parameter G^i_{jk} for a Landsberg space which characterized by the condition

(3.1) $$y_r\, G^{\,r}_{jkh} = -2\, C_{jkh|m}\; y^m = -2\, P_{jkh} = 0\ .$$

Various authors denote the tensor $C_{jkh|m}\, y^m$ by P_{jkh} H. Izumi ([7],[8],[9],[10], [11]), H. Izumi and T.N. Srivastava [12], H. Izumi and M. Yoshida [13] and M. Matsumoto [17].

Remark 3.1. An affinely connected space is necessarily a Landsberg space. However, a Landsberg space need not be an affinely connected space. Hence, any results obtained in anaffinely connected space satisfy in aLandsberg space.

Definition 3.1. If the K^h-birecurrent space is a Landsberg space we called it a K^h*-birecurrent-Landsbergspace* and denoted briefly by K^h-*BR-Landsberg space* .

Now, suppose the K^h-BR-F_n is a Landsberg space .

Remark 3.2. All results in K^h-*BR*-affinely connected space which obtained in the previous section satisfy in K^h-*BR*- Landsberg space .

In section (1), the associate tensor K_{ijhk} of Cartan's fourth curvature tensor K^i_{jkh} satisfing the identity { (4.28),[1] } .

Diff.{ (4.28),[19] } covariantly w.r.t. x^m in the sense of Cartan's, we get

(3.2) $$K_{ijhk|m} + K_{ikjh|m} + K_{ihkj|m} = -2\,(C_{ijs|m}\, H^s_{hk} + C_{ijs}\, H^s_{hk|m} + C_{iks|m}\, H^s_{jh} + C_{iks}\, H^s_{jh|m} + C_{ihs|m} H^s_{kj} + C_{ihs}\, H^s_{kj|m})\ .$$

Diff.(3.2) covariantly w.r.t. x^ℓ in the sense of Cartan and using equ. (1.26), we get

(3.3) $a_{\ell m}(K_{ijhk}+K_{ikjh}+ K_{ihkj}) = -2 (C_{ijs|m|\ell} H^s_{hk}+C_{ijs|m} H^s_{hk|\ell} + C_{ijs|\ell} H^s_{hk|m}$
$+ C_{ijs} H^s_{hk|m|\ell} + C_{iks|m|\ell} H^s_{jh} + C_{iks|m} H^s_{jh|\ell} + C_{iks|\ell} H^s_{jh|m}$
$+ C_{iks} H^s_{jh|m|\ell} + C_{ihs|m|\ell} H^s_{kj} + C_{ihs|m} H^s_{kj|\ell}$
$+ C_{ihs|\ell} H^s_{kj|m}+C_{ihs} H^s_{kj|m|\ell})$.

Transvecting (3.3) by y^m and using (1.17), (3.1) and { (2.6),[1] }, we get

(3.4) $$a_{\ell m} y^m \big(K_{ijhk} + K_{ikjh} + K_{ihkj}\big) = -2\, y^m \big(C_{ijs|\ell} H^s_{hk|m} + C_{iks|\ell} H^s_{jh|m} + C_{ihs|\ell} H^s_{kj|m}\big) - 2\, a_{\ell m}\, y^m \big(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj}\big) .$$

Putting { (4.8),[1] }in (3.4), we get

$$-2\, y^m \big(C_{ijs|\ell} H^s_{hk|m} + C_{iks|\ell} H^s_{jh|m} + C_{ihs|\ell} H^s_{kj|m}\big) = 0$$

or

(3.5) $$C_{ijs|\ell} H^s_{hk|m} + C_{iks|\ell} H^s_{jh|m} + C_{ihs|\ell} H^s_{kj|m} = 0 .$$

Transvecting (3.5) by g^{hi} and using (1.6b) and (1.4b), we get

(3.6) $$C^r_{js|\ell} H^s_{hk|m} + C^r_{ks|\ell} H^s_{jh|m} + C^r_{hs|\ell} H^s_{kj|m} = 0 .$$

Transvecting (3.6) by y^ℓ and using (1.17a), we get

(3.7) $$P^r_{js} H^s_{hk|m} + P^r_{ks} H^s_{jh|m} + P^r_{hs} H^s_{kj|m} = 0 .$$

Transvecting (3.7) by y^h and using (1.7), (1.17b) and (1.21), we get

(3.8) $$P^r_{js} H^s_{k|m} - P^r_{ks} H^s_{j|m} = 0 .$$

Thus, we conclude

Theorem 3.1.*In K^h-BR-Landsberg space, the identities (3.5), (3.6), (3.7) and (3.8)are all hold good.*

Now, transvecting the Bianchi identity (1.9c) for Cartan's third curvature tensor K^i_{jkh} by y^j and using (1.7), (1.12a) and (1.8), we get

(3.9) $$H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + H^s_{hm} P^i_{sk} + H^s_{kh} P^i_{sm} + H^s_{mk} P^i_{sh} = 0 .$$

Diff.(3.9) covariantly w.r.tx^{ℓ} in the sense of Cartan and using equ.(1.27),we get

(3.10) $$a_{\ell m} H^i_{kh} + a_{\ell h} H^i_{mk} + a_{\ell k} H^i_{hm} + H^s_{hm|\ell} P^i_{sk} + H^s_{hm} P^i_{sk|\ell} + H^s_{kh|\ell} P^i_{sm} + H^s_{kh} P^i_{sm|\ell} + H^s_{mk|\ell} P^i_{sh} + H^s_{mk} P^i_{sh|\ell} = 0 .$$

In view of (3.7) the equ.(3.10) can be written as

(3.11) $$a_{\ell m} H^i_{kh} + a_{\ell h} H^i_{mk} + a_{\ell k} H^i_{hm} + H^s_{hm} P^i_{sk|\ell} + H^s_{kh} P^i_{sm|\ell} + H^s_{mk} P^i_{sh|\ell} = 0 .$$

Transvecting (3.11) by y_s and using (1.7)and (1.20b), we get

(3.12) $$\left(a_{\ell m} H^i_{kh} + a_{\ell h} H^i_{mk} + a_{\ell k} H^i_{hm}\right) y_s = 0.$$

which can be written as

(3.13) $$a_{\ell m} H^i_{kh} + a_{\ell h} H^i_{mk} + a_{\ell k} H^i_{hm} = 0 ,$$

provided$y_s \neq 0$, which is equ.(2.26), this emphasizes remark 3.2.

Thus, we conclude

Theorem 3.2. *In K^h-BR-Landsberg space, the identities (3.11),(3.12) and (3.13) are all hold good .*

Transvecting (3.13) by y^m and using (1.7) and (1.21), we get

(3.14) $$\lambda_{\ell} H^i_{kh} = a_{\ell h} H^i_k - a_{\ell k} H^i_h ,$$

since $$a_{\ell m}\, y^m = \lambda_{\ell} ,$$

which can be written as

$$H^i_{kh} = \frac{1}{\lambda_{\ell}}\left(a_{\ell h} H^i_k - a_{\ell k} H^i_h\right)$$

or

(3.15) $$H^i_{kh} = \mu_h H^i_k - \mu_k H^i_h ,$$

where $$\mu_p = \frac{a_{\ell p}}{\lambda_{\ell}} .$$

Diff.(3.15) partially w.r.t . y^j and using(1.18b), we get

(3.16) $$H^i_{jkh} = \dot{\partial}_j\left(\mu_h H^i_k - \mu_k H^i_h\right) .$$

Contracting the indices i and h in (3.15) and using (1.22b) and (1.22c), we get

(3.17) $$H_k = \mu_r\, H^r_k - (n-1)\mu_k\, H .$$

Diff.(3.17) partially w.r.t. y^j and using (1.23a), we get

(3.18) $$H_{jk} = \dot{\partial}_j \,\{\mu_r H^r_k - (n-1)\mu_k\, H\} .$$

Thus, we conclude

Theorem 3.3. *In K^h-BR-Landsberg space, the h(v)-torsion tensor H^i_{kh} , Berwald curvature tensor H^i_{jkh} , the vector H_k and the h-Ricci tensor H_{jk} are defined by the equations (3.15), (3.16), (3.17) and (3.18) respectively*.

References

[1] **Alqufail, M.A.H. ,Qasem**, **F.Y.A.** and **Ali,M.A.A.**:On Study K^h-Birecurrent Finsler Space, The Scientific Journal of The Faculty of Education , Thamar University, to be published.

[2] **Berwald ,L.**: On Finsler and Cartan geometries, III. Two dimensional Finsler spaces with rectilinear externals, Ann. Of Math.(2), 42(1941), 84-122.

[3] **Berwald ,L.**: UberFinslersche und Cartansche Geometric IV. Projectiv- Krummung

allgemeiner affine Raumeskalarer, Krummung, Ann.Math.,(2) 48(1947) 755-781.

[4] **Cartan, É.**:sur les espaces de finsler, C.R .Acad. Sci. (paris), 196 (1933) 582-586.

[5] **Cartan, É.**: *les espaces de finsler, Actualites, Paris*, 79 (1934) 2nd ed (1971) .

[6] **Dwivedi,P.K.**: On *P–Reducible Finsler spaces and applications, Int. Journal of Math.Analysis, Vol. 5, (2011), No. 5, 223-229.

[7] **Izumi, H.** : On *P - *Finsler spaces I, Memo Defence Acad. (Japan)*,16 (1976),133-138.

[8] **Izumi, H.**: Conformal transformations of Finsler spaces I. Concircular transformations of a curve with Finsler metric, Tensor N.S., 31 (1977) 33-41.

[9] **Izumi, H.** : On *P - *Finsler spaces II , Memo . DefenceAcad . (Japan)*,17 (1977)1- 9.

[10] **Izumi, H.**: Conformal transformations of Finsler spaces II. An h-conformal flat Finsler space, Tensor N.S., 34 (1980) 337-359.

[11] **Izumi, H.**: On *P- Finsler space of scalar curvature, Tensor N.S., 38 (1982)

[12] **Izumi, H.** and **Srivastava, T. N.**: On R3-like Finsler space, Tensor N.S., 32(1978) space, Tensor N.S., 34 (1980) 337-359.

[13] **Izumi, H.** and **Yoshida, M.**: On Finsler spaces of perpendicular scalar curvature,Tensor N.S., 32 (1978) 219 - 224.

[14] **Landesberg, G.**: Uber die Krumming in der variation rechnung , Matt. Ann., 65(1908), 313-349.

[15] **Matsumoto, M.**: On h-isotropic and C^h-recurrent Finsler spaces, J. Math. Kyoto Univ . 11 (1971) 1-9.

[16] **Matsumoto, M.**: On Finsler spaces with curvature tensors of some special forms Tensor T.N., 22(1971) 201-204.

[17] **Matsumoto, M.**: Remarks on Berwald and Landsberg Spaces, Contemporary Math. Vol. 195, (1986) 79-81.

[18] **Matsumoto, M.** : Theory of extended point transformation of Finsler space I, Fundermental theorem of projection motions, tensors N.S. vol 45(1987) 109-115.

[19] **Mishra, C. K.** and **GautumLodhi .**: On C^h −recurrent and C^v − recurrent Finsler spaces of second order ,Int. J. Contemp. Math. Sciences, vol.3, (2008), No. 17,801-810.

[20] **Muhib, A.A.A .** : *On independent components of tensor, I-relative tensor and* R^h *–generalized trirecurrent Finder space* , M.Sc . Thesis ,University of Aden . Aden (Yemen) (2009) .

[21] **Pande, H.D** and **Singh, B.**: On existence of the affinely connected Finsler space with recurrent tensor field, Reprinted from Indian Journal of pure & Applied Mathematics,Vol. 8, No.3 (March1977) 295-301.

[22] **Pandey, P.N.**: *Some Problems in Finsler Space*, D.Sc. , thesis University of Allahabad, Allahabad (India), (1993).

[23] **Rund,H.** : *The Differential geometry of Finsler space*, Springer-verlag ,Berlin Götting -Heidelberg,(1959).

[24] **Saleem, A.A.M**.: On Certain Generalized Birecurrent and Trirecurrent Finsler spaces, M.Sc. Thesis, University of Aden, Aden (Yemen) (2011).

[25] **Verma,R**.: *Some transformations in Finsler spaces*, D.Phil. Thesis ,University of Allahabad ,Allahabad (India),1991.

[26] **Yasuda , H** .: On Landsberg spaces , Tensor , N. S . Vol. , 34 (1980) 76-84 .

(1)– ثنائي المعاودة K^h بعض أنواع فضاء فنسلر

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الملخص

ثنائي المعاودة، كما تم R^i_{jkh} تم دراسة فضاء فنسلر الذي يكون فيه الموتر التقوسي ثنائي المعاودة. N^i_{jkh} دراسة فضاء فنسلر الذي يكون فيه الموتر التقوسي العادي ثنائية المعاودة K^i_{jkh} وللاستمرار في دراسة الموترات التقوسية الرابعة لكارتان - ثنائية المعاودة تم k^h وكذلك الموترات الالتوائية ثنائية المعاودة في فضاء فنسلر دراسة بعض الفضاءات واستنتاج بعضها من الأخر في هذه الورقة . كما تم الحصول على مبرهنات مختلفة متعلقة بهذا الفضاء وتم الحصول على متطابقات مختلفة متعلقة بالفضاءات

k^h-birecurrent affinity connected space k^h- bireanrrent Landsbleg space .