كثيرات حدود لاجير المعممة من النوع كيو

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الملخص

في هذه الورقة قدمنا كثيرات حدود لاجير المعممة ذات متغيرومتغيرين من النوع كيو وأيضاً أثبتنا العلاقات التكرارية لهما.

الكلمات المفتاحية : كثيرات حدود لاجير من النوع كيو،الدوال المولدة، العلاقات التكرارية.

The Generalized q-Analogue Laguerre Polynomials

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Abstract

In this paper, we introduce the q-analogue generalized Laguerre polynomials of one variable and two variables. some recurrence relations for these q-polynomials are derived.

Keywords: q-analogue Laguerre polynomials, generating functions, recurrence relations.

1. Introduction

In this section, we will give a summary of the mathematical notations and definitions required in this paper for the convenience of the reader.

The basic hypergeometric or q-hypergeometric function ${}_r\phi_s$ is defined by the series [3]

$$ {}_r\phi_s\left[\begin{matrix} a_1,\ldots,a_r; \\ & q, z \\ b_1,\ldots,b_s; \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r,q)_n}{(b_1,\ldots,b_s,q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)\binom{n}{2}} \frac{z^n}{(q,q)_n}, $$

(1.1)

where $(a_1,\ldots,a_r,q)_n = (a_1;q)_n \ldots (a_r;q)_n$.

Let the q-analogues of Pochhammer symbol or q-shifted factorial be defined by [3]

$$ (a;q)_n = \begin{cases} 1 & , n=0 \\ \prod\limits_{0\le j\le n-1} \left(1-aq^j\right) & , n=1,2,3,\ldots \end{cases} $$

(1.2)

where
$$ \left(q^{-n};q\right)_k = \begin{cases} 0 & k>n \\ \dfrac{(q;q)_n}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk}, & k\le n \end{cases} $$

(1.3)

$$ (0;q)_n = 1, $$

also

$$ (a;q)_{n+k} = (a;q)_n \left(aq^n;q\right)_k, $$

(1.4)

where
$$ \lim_{q\to 1^-} \frac{\left(q^z;q\right)_k}{(1-q)^k} = (z)_k . $$

The q-binomial coefficient is defined by[8]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \quad , \quad 0 \le k \le n, \quad k, n \in N,$$

(1.5)

$$\begin{bmatrix} -n \\ k \end{bmatrix}_q = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(-q^{-n}\right)^k q^{-\binom{k}{2}}, \quad n \in C; k \in N_0 .$$

(1.6)

Exton [2] presented the following q-exponential functions:

$$E(\mu, z; q) = \sum_{n=0}^{\infty} \frac{q^{\mu n(n-1)}}{[n]_q!} z^n ,$$

where $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$.

In Exton's formula, if we replace z by $\frac{x}{1-q}$ and μ by $2a$, we get

$$E\left(2a, \frac{x}{1-q}; q\right) = E_q(x, a),$$

where

$$E_q(x, a) = \sum_{n=0}^{\infty} \frac{q^{a\binom{n}{2}}}{(q;q)_n} x^n ,$$

(1.7)

which satisfies the functional relation

$$E_q(x, a) - E_q(qx, a) = xE_q\left(q^a x, a\right).$$

The above q-function can be rewritten by the formula

$$D_q E_q(x,a) = \frac{1}{1-q} E_q(q^a x, a).$$

(1.8)

The formulas for the q-difference D_q of a addiction, a product and a quotient of functions are [7]

$$D_q(\lambda\ f(x) + \mu\, g(x)) = \lambda\, D_q\, f(x) + \mu\, D_q g(x),$$

(1.9)

$$D_q(\,f(x)g(x)) = f(qx)\,D_q g(x) + g(x)\,D_q\, f(x),$$

(1.10)

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q\, f(x) - f(x)D_q g(x)}{g(x)\,g(qx)}, \quad g(x)\,g(qx) \neq 0$$

(1.11)

Also, the q-analogue of the power (binomial) function $(x \pm y)^n$ is given by [5]

$$(x \pm y)^n = (x \pm y)_n = x^n\left(\mp {y}/{x}\,; q\right)_n = x^n \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} \left(\pm {y}/{x}\right)^k.$$

(1.12)

The Laguerre polynomials $L_n(x)$ of n order are defined by means of a generating relation [6]

$$(1-t)^{-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)\,t^n, \quad |t| < 1,\ 0 < x < \infty$$

(1.13)

and the following series definition

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^k n!\, x^k}{(k!)^2 (n-k)!}.$$

(1.14)

Also, the associated Laguerre polynomials are defined by the generating function [6]

$$(1-t)^{-1-\alpha}\exp\left(\frac{-xt}{1-t}\right)=\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n,$$

(1.15)

and the series definition

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{(-1)^k (1+\alpha)_n\, x^k}{k!(n-k)!(1+\alpha)_k}.$$

(1.16)

The two variable Laguerre polynomials are defined by the generating function [1]:

$$(1-yt)^{-1}\exp\left(\frac{-xt}{1-yt}\right)=\sum_{n=0}^{\infty} L_n(x,y)t^n,$$

(1.17)

Or equivalcutly

$$(1-yt)^{-1-\alpha}\exp\left(\frac{-xt}{1-yt}\right)=\sum_{n=0}^{\infty} L_n^{(\alpha)}(x,y)t^n.$$

(1.18)

The two variables Laguerre polynomials are defined by the series definition

$$L_n(x,y)=n!\sum_{k=0}^{n}\frac{(-1)^k x^k y^{n-k}}{(k!)^2(n-k)!},$$

(1.19)

Further, the two variable associated Laguerre polynomials are defined by [1]

$$L_n^{(\alpha)}(x,y)=\sum_{k=0}^{n}\frac{(-1)^k(1+\alpha)_n x^k y^{n-k}}{k!(n-k)!(1+\alpha)_k}.$$

(1.20)

The q-Laguerre polynomials are defined by [4]

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_1\phi_1\left(\begin{matrix}q^{-n}; \\ \\ q^{\alpha+1};\end{matrix}\quad q;-(1-q)q^{\alpha+n+1}x\right)$$

$$=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(q^{-n};q)_k q^{\binom{k}{2}}(1-q)^k(q^{n+\alpha+1}x)^k}{(q^{\alpha+1};q)_k(q;q)_k},$$

(1.21)

where $\alpha>-1,\ \ 0<q<1$ and $n=0,1,2,3,\ldots.$

The q-Laguerre polynomials are specified by the following generating function:

$$\sum_{n=0}^{\infty}L_n^{(\alpha)}(x;q)t^n=\sum_{n=0}^{\infty}\frac{q^{n^2+\alpha n}(-xt)^n}{\{n\}_q!(t;q)_{1+\alpha+n}}$$

$$=\frac{1}{(t;q)_\infty}{}_1\phi_1\left(\begin{matrix}-x \\ 0\end{matrix}\Big/ q;q^{\alpha+1}t\right),$$

(1.22)

where q-exponential function $e_q(x)$ is defined by

$$\frac{1}{(x;q)_\infty}=\sum_{n\geq 0}\frac{x^n}{(q;q)_n}=e_q(x),$$

(1.23)

and

$$(x;q)_\infty=\sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}\frac{x^n}{(q;q)_n}=E_q(x).$$

(1.24)

The q-Laplace transforms is defined by [9]

$$_qL_s\{f(t)\}=\frac{1}{1-q}\int_0^\infty e_q(-st)\,f(t)\,d(t;q);\qquad R(s)>0$$

(1.25)

and

$$_qL_s\{t^n\}=\frac{(q;q)_n}{s^{n+1}};\quad n>0.$$

(1.26)

2. The Generalized q-Laguerre Polynomials of One Variable

In this section, we introduce the generalized q-analogue Laguerre polynomial of one variable by the following:

$$L_n^{(\alpha)}(x,a;q)=\frac{(q^{\alpha+1};q)_n}{q^{(\alpha+1)n}(q;q)_n}{}_1\phi_1\left(q^{-n},q^{\alpha+1};q^{(a+1)},q^{\alpha+2}x\right).$$

(2.1)

Now, we get generating function of the generalized q-analogue Laguerre polynomials in the form of the following theorem:

Theorem 2.1

The following generating function for the generalized q-analogue Laguerre polynomials $L_n^{(\alpha)}(x,a;q)$ holds true:

$$[1-t]_q^{-1-\alpha} E_q\left[\frac{-xt}{1-t},a\right]=\sum_{n=0}^{\infty} L_n^{(\alpha)}(x,a;q)t^n \quad ,$$

(2.2)

where $|t|<1,\ |q|<1,\ a\in Z^+, 0<x<\infty$.

Proof. Let we denote the left hand sides of (2.2) by W, then

$$W=[1-t]_q^{-1-\alpha} E_q\left[\frac{-xt}{1-t},a\right],$$

and using (1.7), we obtain

$$W=\sum_{k=0}^{\infty}(-1)^k \frac{q^{a\binom{k}{2}}(xt)^k}{(q;q)_k}[1-t]_q^{-1-\alpha-k},$$

(2.3)

by using relation (1.12), we get

$$W=\sum_{k=0}^{\infty}(-1)^k \frac{q^{a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-1-\alpha-k\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-t)^n,$$

(2.4)

and using relation (1.6), we find

$$W=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{q^{a\binom{k}{2}+(-1-\alpha-k)n}(x)^k}{(q;q)_k}\begin{bmatrix}\alpha+k+n\\ n\end{bmatrix}_q t^{n+k},$$

(2.5)

which on using relation (1.5), gives

$$W=\sum_{n=0}^{\infty}\ \sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}+(-1-\alpha-k)n}}{(q;q)_k}\frac{(q;q)_{\alpha+k+n}}{(q;q)_{\alpha+k}(q;q)_n}(x)^k\,t^{n+k}$$

$$=\sum_{n=0}^{\infty}\frac{(q^{\alpha+1};q)_n}{q^{(\alpha+1)n}(q;q)_n}\sum_{k=0}^{n}(-1)^k\frac{q^{\binom{k}{2}-nk}(q;q)_n}{(q;q)_{n-k}}\frac{q^{a\binom{k}{2}+\binom{k}{2}}}{(q^{\alpha+1};q)_k(q;q)_k}(q^{\alpha+2}x)^k\,t^n,$$

from relation (1.3), we get

$$W=\sum_{n=0}^{\infty}\frac{(q^{\alpha+1};q)_n}{q^{(\alpha+1)n}(q;q)_n}\sum_{k=0}^{n}\frac{q^{(a+1)\binom{k}{2}}}{(q;q)_k}\frac{(q^{-n};q)_k}{(q^{\alpha+1};q)_k}(q^{\alpha+2}x)^k\,t^n,$$

(2.6)

now, using definition (2.1), then we obtain the required relation (2.2).

Next, we derive some recurrence relations for the polynomials $L_n^{(\alpha)}(x,a\,;q)$ in the form of the following theorems:

Theorem 2.2

The generalized q-analogue Laguerre polynomials of one variable $L_n^{(\alpha)}(x,a;q)$ satisfy the following relation:

$$D_x L_n^{(\alpha)}(x,a;q)=-L_{n-1}^{(\alpha+1)}(q^a x,a;q).$$

(2.7)

Proof.

Differentiating both sides of (2.2) with respect to x, we get

$$\sum_{n=0}^{\infty} D_x L_n^{(\alpha)}(x,a;q)t^n = -\frac{t}{(1-q)}\sum_{k=0}^{\infty}(-1)^k \frac{q^{ak+a\binom{k}{2}}(xt)^k}{(q;q)_k}[1-t]_q^{-2-\alpha-k},$$

which on using (1.12), becomes

$$\sum_{n=0}^{\infty} D_x L_n^{(\alpha)}(x,a;q)t^n = -t\sum_{k=0}^{\infty}(-1)^k \frac{q^{ak+a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix} -2-\alpha-k \\ n \end{bmatrix}_q q^{\binom{n}{2}}(-t)^n,$$

(2.8)

by using relation (1.6), we obtain

$$\sum_{n=0}^{\infty} D_x L_n^{(\alpha)}(x,a;q)t^n = -t\sum_{k=0}^{\infty}(-1)^k \frac{q^{ak+a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix} \alpha+k+n+1 \\ n \end{bmatrix}_q \left(q^{-2-\alpha-k}\right)^n t^n$$

$$= -\sum_{n=0}^{\infty} \frac{\left(q^{\alpha+2};q\right)_{n-1}}{q^{(\alpha+2)(n-1)}(q;q)_{n-1}}\sum_{k=0}^{n-1} q^{ak+a\binom{k}{2}+\binom{k}{2}} \frac{(-1)^k q^{\binom{k}{2}-(n-1)k}(q;q)_{n-1}\left(q^{\alpha+3}x\right)^k}{(q;q)_k\left(q^{\alpha+2};q\right)_k(q;q)_{n-k-1}}\, t^n,$$

on using relation (1.5), we find

$$\sum_{n=0}^{\infty} D_x L_n^{(\alpha)}(x,a;q)t^n$$

$$= -\sum_{n=0}^{\infty} \frac{\left(q^{\alpha+2};q\right)_{n-1}}{q^{(\alpha+2)(n-1)}(q;q)_{n-1}}\sum_{k=0}^{n-1} q^{ak+a\binom{k}{2}+\binom{k}{2}} \frac{(-1)^k q^{\binom{k}{2}-(n-1)k}(q;q)_{n-1}\left(q^{\alpha+3}x\right)^k}{(q;q)_k\left(q^{\alpha+2};q\right)_k(q;q)_{n-k-1}}\, t^n,$$

Using relation (1.3), we get

$$\sum_{n=0}^{\infty} D_x L_n^{(\alpha)}(x,a;q)t^n = -\sum_{n=0}^{\infty} \frac{\left(q^{\alpha+2};q\right)_{n-1}}{q^{(\alpha+2)(n-1)}(q;q)_{n-1}} \sum_{k=0}^{n-1} q^{ak+a\binom{k}{2}+\binom{k}{2}} \frac{\left(q^{1-n};q\right)_k \left(q^{\alpha+3}x\right)^k}{(q;q)_k \left(q^{\alpha+2};q\right)_k} t^n$$

$$= -\frac{\left(q^{\alpha+2};q\right)_{n-1}}{q^{(\alpha+2)(n-1)}(q;q)_{n-1}} {}_1\phi_1\left(q^{1-n}, q^{\alpha+2}; q^{a+1}, q^{a+\alpha+3}x\right).$$

By equating the coefficients of t^n, we obtain the required relation (2.7).

Theorem 2.3

The generalized q-analogue Laguerre polynomials of one variable $L_n^{(\alpha)}(x,a;q)$ satisfy the following relation:

$$[n+1]_q L_{n+1}^{(\alpha)}(x,a;q) = (\alpha+1) L_n^{(\alpha+1)}(x,a;q)$$

$$-x\sum_{k=0}^{n} \sum_{r=0}^{n-k} \frac{(-1)^k q^{(n-k-r)(-1-k)} \left(q^{k+1};q\right)_{n-k-r} \left(q^{\alpha+2};q\right)_r x^k}{q^{(2+\alpha)r} (q;q)_k (q;q)_{n-k-r} (q;q)_r}.$$

(2.9)

Proof.

Differentiating both sides of (2.2) with respect to t and using relation (1.10), we get

$$\sum_{n=0}^{\infty} [n]_q L_n^{(\alpha)}(x,a;q)t^{n-1} = (\alpha+1)\sum_{k=0}^{\infty}(-1)^k \frac{q^{a\binom{k}{2}}(xt)^k}{(q;q)_k}[1-t]_q^{-2-\alpha-k}$$

$$-\frac{x}{(1-q)}[1-qt]_q^{-2-\alpha}\sum_{k=0}^{\infty}(-1)^k \frac{q^{ak+a\binom{k}{2}}(xt)^k}{(q;q)_k}[1-t]_q^{-1-k},$$

(2.10)

using the relation (1.12), in the above equation we get

$$\sum_{n=0}^{\infty}[n+1]_q L_{n+1}^{(\alpha)}(x,a;q)t^n=(\alpha+1)\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-2-\alpha-k\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-t)^n$$

$$-\frac{x}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}-2-\alpha\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-qt)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-1-k\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-t)^n,$$

by using relation (1.6), we find

$$\sum_{n=0}^{\infty}[n+1]_q L_{n+1}^{(\alpha)}(x,a;q)t^n=(\alpha+1)\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(x)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha+k+n+1\\ n\end{bmatrix}_q\left(q^{-2-\alpha-k}\right)^n t^{n+k}$$

$$-\frac{x}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}\alpha+r+1\\ r\end{bmatrix}_q\left(q^{-2-\alpha}\right)^r(qt)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(x)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}k+n\\ n\end{bmatrix}_q\left(q^{-1-k}\right)^n t^{n+k},$$

which on using relation (1.5) gives

$$\sum_{n=0}^{\infty}[n+1]_q L_{n+1}^{(\alpha)}(x,a;q)t^n=(\alpha+1)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k q^{a\binom{k}{2}+(-2-\alpha-k)(n-k)}\left(q^{\alpha+2};q\right)_n(x)^k}{(q;q)_k\left(q^{\alpha+2};q\right)_k(q;q)_{n-k}}t^n$$

$$-x\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^k q^{(-2-\alpha)r+(-1-k)n}(q;q)_{k+n}(q;q)_{\alpha+r+1}(x)^k}{(q;q)_k(q;q)_k(q;q)_n(q;q)_{\alpha+1}(q;q)_r}t^{n+k+r},$$

(2.11)

applying relation (1.4), we get

$$\sum_{n=0}^{\infty}[n+1]_q L_{n+1}^{(\alpha)}(x,a;q)t^n=(\alpha+1)\sum_{n=0}^{\infty}\frac{(q^{\alpha+2};q)_n}{q^{(2+\alpha)n}(q\ ;q)_n}\sum_{k=0}^{n}q^{(a+1)\binom{k}{2}}\frac{(-1)^k q^{\binom{k}{2}-nk}(q\ ;q)_n(q^{\alpha+3}x)^k}{(q;q)_k(q^{\alpha+2};q)_k(q;q)_{n-k}}t^n$$
$$-x\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}\frac{(-1)^k q^{(n-k-r)(-1-k)}(q^{k+1};q)_{n-k-r}(q^{\alpha+2};q)_r(x)^k}{q^{(2+\alpha)r}(q;q)_k(q;q)_{n-k-r}(q;q)_r}t^n,$$

Using relation (1.3), we get

$$[n+1]_q L_{n+1}^{(\alpha)}(x,a;q)=(\alpha+1)\frac{(q^{\alpha+2};q)_n}{q^{(2+\alpha)n}(q\ ;q)_n}\sum_{k=0}^{n}q^{(a+1)\binom{k}{2}}\frac{(q^{-n};q)_k(q^{\alpha+3}x)^k}{(q;q)_k(q^{\alpha+2};q)_k}$$

$$-x\sum_{k=0}^{n}\sum_{r=0}^{n-k}\frac{(-1)^k q^{(n-k-r)(-1-k)}(q^{k+1};q)_{n-k-r}(q^{\alpha+2};q)_r(x)^k}{q^{(2+\alpha)r}(q;q)_k(q;q)_{n-k-r}(q;q)_r}$$
$$=(\alpha+1)\frac{(q^{\alpha+2};q)_n}{q^{(2+\alpha)n}(q\ ;q)_n}{}_1\phi_1(q^{-n},q^{\alpha+2};q^{a+1},q^{\alpha+3}x)$$
$$-x\sum_{k=0}^{n}\sum_{r=0}^{n-k}\frac{(-1)^k q^{(n-k-r)(-1-k)}(q^{k+1};q)_{n-k-r}(q^{\alpha+2};q)_r x^k}{q^{(2+\alpha)r}(q;q)_k(q;q)_{n-k-r}(q;q)_r},$$

By using definition (2.1), we get the required relation (2.9).

Theorem 2.4

The q-Laplace transform of generalized q- Laguerre polynomials $L_n^{(\alpha)}(x,a;q)$ is defined by

$$_qL_s\{L_n^{(\alpha)}(x,a;q)\}=\frac{1}{1-q}\frac{(q^{\alpha+1};q)_n}{sq^{(\alpha+1)n}(q;q)_n}\sum_{k=0}^{n}q^{(a+1)\binom{k}{2}}\frac{(q^{-n};q)_k}{(q^{\alpha+1};q)_k}\left(\frac{q^{(\alpha+2)}}{s}\right)^k,$$

(2.12)

where $t\geq 0$.

Proof

By using definition of the q-Laplace transform (1.25) then the relation (2.2) can be written as

$$
{}_qL_s\left\{L_n^{(\alpha)}(x,a;q)\right\}=\frac{1}{1-q}\int_0^{\infty}e_q(-st)\left\{\frac{\left(q^{\alpha+1};q\right)_n}{q^{(\alpha+1)n}(q;q)_n}\sum_{k=0}^{n}\frac{q^{(a+1)\binom{k}{2}}}{(q;q)_k}\frac{\left(q^{-n};q\right)_k}{\left(q^{\alpha+1};q\right)_k}\left(q^{\alpha+2}t\right)^k\right\}d(t;q)
$$

$$
=\frac{1}{1-q}\frac{\left(q^{\alpha+1};q\right)_n}{q^{(\alpha+1)n}(q;q)_n}\sum_{k=0}^{n}\frac{q^{(a+1)\binom{k}{2}}}{(q;q)_k}\frac{\left(q^{-n};q\right)_k}{\left(q^{\alpha+1};q\right)_k}\left(q^{\alpha+2}\right)^k{}_qL_s\left\{t^k\right\}
$$

By using relation (1.26), we get

$$
{}_qL_s\left\{L_n^{(\alpha)}(x,a;q)\right\}=\frac{1}{1-q}\frac{\left(q^{\alpha+1};q\right)_n}{q^{(\alpha+1)n}(q;q)_n}\sum_{k=0}^{n}\frac{q^{(a+1)\binom{k}{2}+(\alpha+2)k}}{(q;q)_k}\frac{\left(q^{-n};q\right)_k}{\left(q^{\alpha+1};q\right)_k}\left\{\frac{(q;q)_k}{s^{k+1}}\right\},
$$

which is the required relation (2.12).

3. The Generalized q-Laguerre Polynomials of Two Variables

We introduce the q-Laguerre polynomials of two variables by the following:

$$
L_n^{(\alpha)}(x,y,a;q)=\frac{\left(q^{\alpha+1};q\right)_n y^n}{q^{(\alpha+1)n}(q;q)_n}{}_1\phi_1\left(q^{-n},q^{\alpha+1};q^{(a+1)},q^{\alpha+2}\frac{x}{y}\right).
$$

(3.1)

Now, we derive the generating function for $L_n^{(\alpha)}(x,y,a;q)$ in the form of the following theorem:

Theorem 3.1

The following generating function for the q-analogue Laguerre polynomials $L_n^{(\alpha)}(x,y,a;q)$ holds true:

$$[1-yt]_q^{-1-\alpha} E_q\left[\frac{-xt}{1-yt}, a\right] = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x, y, a; q) t^n, \ a \in Z^+, \ |t| < 1, \ |q| < 1 .$$

(3.2)

Proof. Let we denote the left hand sides of (3.2) by H, then using (1.7), we obtain

$$H = [1-yt]_q^{-1-\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k q^{a\binom{k}{2}} (xt)^k [1-yt]_q^{-k}}{(q;q)_k},$$

by using the relation (1.12), we get

$$H = \sum_{k=0}^{\infty} (-1)^k \frac{q^{a\binom{k}{2}} (xt)^k}{(q;q)_k} \sum_{n=0}^{\infty} \begin{bmatrix} -1-\alpha-k \\ n \end{bmatrix}_q q^{\binom{n}{2}} (-yt)^n ,$$

(3.3)

applying relation (1.6),we obtain

$$H = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{a\binom{k}{2}} (x)^k}{(q;q)_k} \begin{bmatrix} \alpha+k+n \\ n \end{bmatrix}_q \left(q^{(-1-\alpha-k)} y\right)^n t^{n+k},$$

From relation (1.5), we find

$$H = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1};q)_n y^n}{q^{(\alpha+1)n} (q;q)_n} \sum_{k=0}^{n} (-1)^k \frac{q^{(a+1)\binom{k}{2}+\binom{k}{2}-nk}}{(q;q)_k} \frac{(q;q)_n}{(q^{\alpha+1};q)_k (q;q)_{n-k}} \left(q^{\alpha+2} \frac{x}{y}\right)^k t^n ,$$

(3.4)

using the relation (1.3), we obtain

$$H = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1};q)_n y^n}{q^{(\alpha+1)n} (q;q)_n} \sum_{k=0}^{\infty} \frac{q^{(a+1)\binom{k}{2}}}{(q;q)_k} \frac{(q^{-n};q)_k}{(q^{\alpha+1};q)_k} \left(q^{\alpha+2} \frac{x}{y}\right)^k t^n ,$$

which by using definition (3.1) yields the required relation (3.2).

Note.

$$L_n^{(\alpha)}(x,1,a\,;q)=L_n^{(\alpha)}(x,a\,;q),\qquad L_n^{(\alpha)}(x,y,0\,;q)=L_n^{(\alpha)}(x,y;q),$$

and

$$L_n^{(\alpha)}(x,-y;q)=(-1)^n\,L_n^{(\alpha)}(x,y;q).$$

Next, we obtain some recurrence relations in the form of the following theorems:

Theorem 3.2

The generalized q-analogue Laguerre polynomials of two variable $L_n^{(\alpha)}(x,y,a;q)$ satisfy the following relations:

$$\frac{\partial}{\partial x}L_n^{(\alpha)}(x,y,a;q)=-\frac{1}{(1-q)}L_{n-1}^{(\alpha+1)}(q^a x,y,a;q),$$

(3.5)

and

$$\frac{\partial}{\partial y}L_n^{(\alpha)}(x,y,a;q)t^n=(\alpha+1)L_{n-1}^{(\alpha+1)}(x,y,a;q)$$

$$-\frac{x}{(1-q)}\sum_{k=0}^{n-2}\sum_{r=0}^{n-k-2}(-1)^k\frac{q^{a\binom{k}{2}}(q^{\alpha+2};q)_r(q^{k+1};q)_{n-k-r-2}(x)^k\,y^{n-k-r-2}}{q^{(2+\alpha)r+(1+k)(n-k-r-2)}(q;q)_r(q\;;q)_k(q;q)_{n-k-r-2}}.$$

(3.6)

Proof.

Differentiating both sides of (3.2) with respect to x, we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_n^{(\alpha)}(x, y, a; q) t^n = [1 - yt]_q^{-1-\alpha} \left\{ \frac{-t}{(1-q)[1-yt]_q} \right\} E_q\left[-\frac{q^a xt}{1-yt}, a \right]$$

$$= -\frac{t}{(1-q)} \sum_{k=0}^{\infty} (-1)^k \frac{q^{ak+a\binom{k}{2}} (xt)^k}{(q;q)_k} [1-yt]_q^{-2-\alpha-k},$$

(3.7)

by using relation (1.6), we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_n^{(\alpha)}(x, y, a; q) t^n = -\frac{t}{(1-q)} \sum_{k=0}^{\infty} (-1)^k \frac{q^{ak+a\binom{k}{2}} (xt)^k}{(q;q)_k} \sum_{n=0}^{\infty} \begin{bmatrix} -2-\alpha-k \\ n \end{bmatrix}_q q^{\binom{n}{2}} (-yt)^n$$

,

applying relation (1.6), we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_n^{(\alpha)}(x, y, a; q) t^n = -\frac{t}{(1-q)} \sum_{k=0}^{\infty} (-1)^k \frac{q^{ak+a\binom{k}{2}} (xt)^k}{(q;q)_k} \sum_{n=0}^{\infty} \begin{bmatrix} \alpha+k+n+1 \\ n \end{bmatrix}_q q^{(-2-\alpha-k)n} (yt)^n,$$

which by using relation (1.5), we find

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_n^{(\alpha)}(x, y, a; q) t^n$$

$$= -\frac{1}{(1-q)} \sum_{n=0}^{\infty} \frac{(q^{\alpha+2};q)_{n-1} y^{n-1}}{q^{(\alpha+2)(n-1)} (q;q)_{n-1}} \sum_{k=0}^{n-1} (-1)^k \frac{q^{ak+(a+1)\binom{k}{2}+\binom{k}{2}-(n-1)k} (q;q)_{n-1}}{(q;q)_k (q^{\alpha+2};q)_k (q;q)_{n-k-1}} \left(q^{\alpha+3} \frac{x}{y} \right)^k t^n$$

$$= -\frac{1}{(1-q)} \sum_{n=0}^{\infty} \frac{(q^{\alpha+2};q)_{n-1} y^{n-1}}{q^{(\alpha+2)(n-1)} (q;q)_{n-1}} \sum_{k=0}^{n-1} \frac{q^{ak+(a+1)\binom{k}{2}} (q^{1-n};q)_k}{(q;q)_k (q^{\alpha+2};q)_k} \left(q^{\alpha+3} \frac{x}{y} \right)^k t^n,$$

(3.8)

which on equating the coefficient of t^n yields the required relation (3.5).

Again, differentiating the both sides (3.2) with respect to y, we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial y} L_n^{(\alpha)}(x,y,a;q)t^n = (\alpha+1)t[1-yt]_q^{-2-\alpha} E_q\left[-\frac{xt}{1-yt},a\right]$$

$$+[1-qyt]_q^{-2-\alpha}\left[\frac{-xt^2}{(1-q)[1-yt]}\right]E_q\left[-\frac{q^a xt}{1-yt},a\right],$$

(3.9)

By using relations (1.7) and (1.9), we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial y} L_n^{(\alpha)}(x,y,a;q)t^n = (\alpha+1)t\sum_{k=0}^{\infty}(-1)^k \frac{q^{a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-2-\alpha-k\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-yt)^n$$

$$-\frac{xt^2}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}-2-\alpha\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-qyt)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-1-k\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-yt)^n$$

,

and using relation (1.6), we find

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial y} L_n^{(\alpha)}(x,y,a;q)t^n = (\alpha+1)t\sum_{k=0}^{\infty}(-1)^k \frac{q^{a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha+k+n+1\\ n\end{bmatrix}_q q^{(-2-\alpha-k)n}(yt)^n$$

$$-\frac{xt^2}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}\alpha+r+1\\ r\end{bmatrix}_q q^{(-2-\alpha)r}(qyt)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}k+n\\ n\end{bmatrix}_q q^{(-1-k)n}(yt)^n$$

,

By using relation (1.5), we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial y} L_n^{(\alpha)}(x,y,a;q)t^n$$

$$= (\alpha+1)\sum_{n=0}^{\infty} \frac{(q^{\alpha+2};q)_{n-1}\, y^{n-1}}{q^{(\alpha+2)(n-1)}(q;q)_{n-1}} \sum_{k=0}^{n-1} (-1)^k \frac{q^{(a+1)\binom{k}{2}+\binom{k}{2}-(n-1)k}(q;q)_{n-1}}{(q;q)_k (q^{\alpha+2};q)_k (q;q)_{n-k-1}} \left(\frac{q^{\alpha+3}x}{y}\right)^k t^n$$

$$-\frac{x}{(1-q)}\sum_{n=0}^{\infty}\ \sum_{k=0}^{n-2}\ \sum_{r=0}^{n-k-2} (-1)^k \frac{q^{a\binom{k}{2}}(q^{\alpha+2};q)_r (q^{k+1};q)_{n-k-r-2}(x)^k\, y^{n-k-r-2}}{q^{(2+\alpha)r+(1+k)(n-k-r-2)}(q;q)_r (q\ ;q)_k (q;q)_{n-k-r-2}}\, t^n,$$

(3.10)

which on using relation (1.3), we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial y} L_n^{(\alpha)}(x,y,a;q)t^n$$

$$= (\alpha+1)\sum_{n=0}^{\infty} \frac{(q^{\alpha+2};q)_{n-1}\, y^{n-1}}{q^{(\alpha+2)(n-1)}(q;q)_{n-1}} \sum_{k=0}^{n-1} \frac{q^{(a+1)\binom{k}{2}}(q^{1-n};q)_k}{(q;q)_k (q^{\alpha+2};q)_k} \left(\frac{q^{\alpha+3}x}{y}\right)^k t^n$$

$$-\frac{x}{(1-q)}\sum_{n=0}^{\infty}\ \sum_{k=0}^{n-2}\ \sum_{r=0}^{n-k-2} (-1)^k \frac{q^{a\binom{k}{2}}(q^{\alpha+2};q)_r (q^{k+1};q)_{n-k-r-2}(x)^k\, y^{n-k-r-2}}{q^{(2+\alpha)r+(1+k)(n-k-r-2)}(q;q)_r (q\ ;q)_k (q;q)_{n-k-r-2}}\, t^n,$$

then equating the coefficient of t^n yields the required relation (3.6).

Theorem 3.3

The generalized q-analogue Laguerre polynomials of two variable $L_n^{(\alpha)}(x,y,a;q)$ satisfy the following relations:

$$\sum_{n=0}^{\infty} [n+1]_q\, L_{n+1}^{(\alpha)}(x,y,a;q)t^n = (\alpha+1)\frac{(q^{\alpha+2};q)_n\, y^{n+1}}{q^{(\alpha+2)n}(q;q)_n}\, {}_1\phi_1\left(q^{-n}, q^{\alpha+1}; q^{(a+1)}, q^{\alpha+3}\frac{x}{y}\right)$$

$$-\frac{x(y)^n}{(1-q)}\sum_{k=0}^{n}\sum_{k.r=0}^{n-k}(-1)^k\frac{q^{ak+a\binom{k}{2}}(q^{\alpha+2};q)_r(q^{k+1};q)_{n-k-r}}{q^{(2+\alpha)r+(1+k)(n-k-r)}(q;q)_r(q;q)_k(q;q)_{n-k-r}}\left(\frac{x}{y}\right)^k.$$

(3.11)

Proof.

Differentiating both sides of (3.2) with respect to t, we get

$$\sum_{n=0}^{\infty}[n]_q L_n^{(\alpha)}(x,y,a;q)t^{n-1}=(\alpha+1)y[1-yt]_q^{-2-\alpha}E_q\left[-\frac{xt}{1-yt},a\right]$$
$$+\left\{\frac{-x}{(1-q)(1-yt)(1-qyt)}\right\}[1-qyt]_q^{-1-\alpha}E_q\left[-\frac{q^a xt}{1-yt},a\right],$$

by using relation (1.9), we get

$$\sum_{n=0}^{\infty}[n+1]_q L_{n+1}^{(\alpha)}(x,y,a;q)t^n=(\alpha+1)y\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-2-\alpha-k\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-yt)^n$$
$$-\frac{x}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}-2-\alpha\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-qyt)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-1-k\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-yt)^n,$$

by using relations (1.6), we get

$$\sum_{n=0}^{\infty}[n+1]_q L_{n+1}^{(\alpha)}(x,y,a;q)t^n$$
$$=(\alpha+1)y\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(x)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}\alpha+k+n+1\\ n\end{bmatrix}_q q^{(-2-\alpha-k)n}y^n t^{n+k}$$
$$-\frac{x}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}\alpha+r+1\\ r\end{bmatrix}_q q^{(-2-\alpha)r}(qyt)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}k+n\\ n\end{bmatrix}_q q^{(-1-k)n}(yt)^n,$$

which using relations (1.6), we find

$$\sum_{n=0}^{\infty}[n+1]_q L_{n+1}^{(\alpha)}(x,y,a;q)t^n=(\alpha+1)\sum_{n,k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}+(-2-\alpha-k)n}(q;q)_{\alpha+1+k+n}(x)^k y^{n+1}}{(q;q)_k(q;q)_{\alpha+1+k}(q;q)_n}t^{n+k}$$

$$-\frac{x}{(1-q)}\sum_{k=0}^{\infty}\sum_{k.r=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}+(-2-\alpha)r+(-1-k)n}(q;q)_{\alpha+r+1}(q;q)_{k+n}(x)^k(y)^{n+r}}{(q;q)_{\alpha+1}(q;q)_r(q;q)_k(q;q)_k(q;q)_n}t^{n+k+r},$$

(3.12)

Hence,

$$\sum_{n=0}^{\infty}[n+1]_q L_{n+1}^{(\alpha)}(x,y,a;q)t^n$$

$$=(\alpha+1)y\sum_{n=0}^{\infty}\frac{(q^{\alpha+2};q)_n y^n}{q^{(\alpha+2)n}(q;q)_n}\sum_{k=0}^{n}(-1)^k\frac{q^{(a+1)\binom{k}{2}+\binom{k}{2}-nk}(q;q)_n}{(q;q)_k(q^{\alpha+1};q)_k(q;q)_{n-k}}\left(q^{\alpha+3}\frac{x}{y}\right)^k t^n$$

$$-\frac{x}{(1-q)}\sum_{n=0}^{\infty}(y)^n\sum_{k=0}^{n}\sum_{k.r=0}^{n-k}(-1)^k\frac{q^{ak+a\binom{k}{2}}(q^{\alpha+2};q)_r(q^{k+1};q)_{n-k-r}}{q^{(2+\alpha)r+(1+k)(n-k-r)}(q;q)_r(q;q)_k(q;q)_{n-k-r}}\left(\frac{x}{y}\right)^k t^n$$

Also, using relation (1.3), we obtain

$$\sum_{n=0}^{\infty}[n+1]_q L_{n+1}^{(\alpha)}(x,y,a;q)t^n=(\alpha+1)y\sum_{n=0}^{\infty}\frac{(q^{\alpha+2};q)_n y^{n+1}}{q^{(\alpha+2)n}(q;q)_n}\sum_{k=0}^{n}\frac{q^{(a+1)\binom{k}{2}}(q^{-n};q)_k}{(q;q)_k(q^{\alpha+1};q)_k}\left(q^{\alpha+3}\frac{x}{y}\right)^k t^n$$

$$-\frac{x}{(1-q)}\sum_{n=0}^{\infty}(y)^n\sum_{k=0}^{n}\sum_{k.r=0}^{n-k}(-1)^k\frac{q^{ak+a\binom{k}{2}}(q^{\alpha+2};q)_r(q^{k+1};q)_{n-k-r}}{q^{(2+\alpha)r+(1+k)(n-k-r)}(q;q)_r(q;q)_k(q;q)_{n-k-r}}\left(\frac{x}{y}\right)^k t^n$$

Which on equating the coefficients of t^n yields the required relation (3.11).

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