Some New Results On Fuzzy Soft Spaces

S. Saleh and Amani AL-Salemi
Mathematics Department, Faculty of Education-Zabid, Hodeidah University, Yemen.

Abstract. The main aim of this paper is to construct some fuzzy soft topologies from other topological structures namely, crisp topology, soft topology, and fuzzy topology and vice versa. For this reason, we defined
and studied some notions in this sequel. Some results and properties of these notions are investigated. Finally, the relationships between these topological structures are presented with some necessary examples.

Keywords: Fuzzy set, soft set, fuzzy soft set, soft characteristic function, soft support, soft quasi-coincident, soft topology, fuzzy soft topology.

1. Introduction.

There are some mathematical tools for dealing with uncertainties two of them are fuzzy set theory, introduced by Zadeh [24], and soft set theory developed by Molodtsov [16]. The soft set theory has been applied in many fields [1-3,7-9,13,15,21,23,25]. Fuzzy soft set, which is a combination of fuzzy set and soft set was introduced by Maji et al.[14]. Then, Tanay et al. [22] introduced topological structure of fuzzy soft sets and gave some basic properties of it by following Chang [5]. Also, Roy and Samanta [20] gave the definition of fuzzy soft topology over the initial universe set. Fuzzy soft sets and its applications have been studied in recent time [2-4,6,8,10,17,19].

In this paper, we define and study some new concepts and properties related to fuzzy soft spaces. This work is an attempt to introduce the notions of soft characteristic function of crisp and soft set on X, soft support of fuzzy soft sets, and for the concept of soft quasi-coincidence more properties are given. According to these notions we established some new results and relations on fuzzy soft spaces, we show how some fuzzy soft topologies are derived from a crisp, soft, and fuzzy topology and vice versa. Finally, the relationships between these topological structures are presented with some necessary examples.

2. Preliminaries

Here are some definitions and results required in the sequel. Throughout this work, X refers to an initial universe, E be the set of all parameters for X, $P(X)$ is the power set of X and I^X (where, $I = [0,1]$) be the set of all fuzzy subsets of X.

Definition 2.1 [11]
A crisp set $A \subseteq X$ is a set characterized by the function $\chi_A : X \longrightarrow \{0,1\}$ called the characteristic function and A can be represented as follows,
$A = \{x \in X : \chi_A(x) = 1 \ if \ x \in A \ , \ \chi_A(x) = 0 \ if \ \ x \notin A \ \}$.

Correspondence author: S. Saleh (s_wosabi@yahoo.com)

Definition 2.2 [11,24]
A fuzzy set A of X is a set characterized by the membership function $A : X \longrightarrow I$ and A can be represented by ordered pairs $A = \{(x, A(x)) : x \in X, A(x) \in I\}$, $A(x)$ represents the degree of membership of x in A for $x \in X$.
A fuzzy point x_λ ($\lambda \in (0,1]$) is a fuzzy set in X given by, $x_\lambda(y) = \lambda$ at $x = y$ and $x_\lambda(y) = 0$ otherwise $\forall \ y \in X$. For $\alpha \in I$, $\underline{\alpha} \in I^X$ refers to the fuzzy constant function where, $\underline{\alpha}(x) = \alpha \ \ \forall \ x \in X$. The support of $A \in I^X$ is the crisp set given by, $S(A) = \{x \in X : A(x) > 0\}$. For $A , B \in I^X$ and $A_i \subseteq I^X, i \in J$ we have:
$S(A \cap B) = S(A) \cap S(B)$, $S\left(\underset{i\in J}{\cup} A_i\right) = \underset{i\in J}{\cup} S(A_i)$, $S(\underline{0}) = \emptyset$, $S(\underline{1}) = X$ and $S(\chi_A) = A$

For $A, B \in I^X$, the basic operations for fuzzy sets are given by Zadeh [24] as:
(1) $A \subseteq B \Leftrightarrow A(x) \leq B(x) \ \forall x \in X$.
(2) $A = B \Leftrightarrow A(x) = B(x) \ \forall x \in X$.
(3) $C = A \cup B \Leftrightarrow C(x) = A(x) \vee B(x) \ \forall x \in X$.

(4) $D = A \cap B \Leftrightarrow \mu_D(x) = A(x) \wedge B(x)\ \forall x \in X.$
(5) $H = A^c \Leftrightarrow A^c(x) = 1 - A(x)\ \forall x \in X.$

Definition 2.2 [12,16]
A soft set $F_E = (F, E)$ on X is a mapping $F: E \longrightarrow P(X)$ where, the value $F(e)$ is a set called e-element of the soft set for all $e \in E$. It is worth noting that the set $F(e) \in P(X)$ may be arbitrary, empty or have nonempty intersection. Thus a soft set over X can be represented by the set of ordered pairs $F_E = \{(e, F(e)) :\ e \in E\ ,$ $F(e) \in P(X)\}$. The family of all soft sets over X is denoted by $SS(X, E)$.

Definition 2.3 [12,15,16]
Let $F_E = (F, E) \in SS(X, E)$ be a soft set over X. Then:
i) The soft set F_E is called a null, denoted by $\emptyset_E$, if $F(e) = \emptyset\ \forall\ e \in E$, and
if $F(e) = X\ \ \forall e \in E$, then F_E is called an absolute soft set, denoted by X_E.
ii) If $F(e) = \{x\}$ and $F(e') = \emptyset$ for all $e' \in E - \{e\}$, then F_E is called a soft
point in (X, E) and denoted by x_e . The complement of a soft point x_e is a
soft set over X denoted by x_e^c and given by, $x_e^c(e) = X - \{x\}$, $x_e^c(e') = X$ for
all $e' \in E - \{e\}$. The soft point $x_e \,\tilde{\in}\, F_E$ iff for the element $e \in E,\ x \in F(e)$.
The set of all soft points over X is denoted by $SP(X, E)$.
iii) The complement of F_E denoted by F_E^c where $F^c: E \longrightarrow P(X)$ is a mapping
given by, $F^c(e) = X - F(e)$ for all $e \in E$. Clearly $(F_E^c)^c = F_E$.
vi) The union of F_E and G_E is a soft set H_E given by $H(e) = F(e) \cup G(e)$
for all $e \in E$ and denoted by, $F_E \,\tilde{\cup}\, G_E$.
v) The intersection of F_E and G_E is a soft set H_E defined by,
$H(e) = F(e) \cap G(e)$ for all $e \in E$ and denoted by $F_E \,\tilde{\cap}\, G_E$.

Definition 2.4 [15,25]
A soft topological space is the triple (X, τ^*, E) where X an universe set, E is
the set of parameters and τ^* is the collection of soft sets over X satisfies:
$i)$ $\emptyset_E$, $X_E \in \tau^*$,
$ii)$ if F_E , $G_E \in \tau^*$, then $F_E \ \tilde{\cap} \ G_E \in \tau^*$,
$iii)$ if $F_{iE} \in \tau^*$ $\forall \ i \in J$, then $\tilde{\cup}_{i \in J} F_{iE} \in \tau^*$.

The members of τ^* are called soft open sets in X and denoted by, $SO(X, \tau^*, E)$. A soft set F_E over X is called soft closed in X iff $F_E^c \in \tau^*$ and denoted by $SC(X, \tau^*, E)$.

Definition 2.5 [14,20]
A fuzzy soft set $f_E = (f, E)$ over X with the set E of parameters is defined by the set of ordered pairs $f_E = \{(e, f(e)) : e \in E , f(e) \in I^X\}$. Here f is a mapping given by $f: E \longrightarrow I^X$ and the value $f(e)$ is a fuzzy set called e-element of the fuzzy soft set for all $e \in E$. The family of all fuzzy soft sets over X is denoted by $FSS(X, E)$.

Definition 2.6 [2,14,20]
Let $f_E, g_E \in FSS(X, E)$ are two fuzzy soft sets over X.Then:
i) The fuzzy soft set f_E is called a null fuzzy soft set, denoted by 0_E, if $f(e) = \underline{0}$ for every $e \in E$ where, $\underline{0}(x) = 0 \ \forall \ x \in X$.
ii) If $f(e) = \underline{1}$ for every $e \in E$, then f_E is called an universal fuzzy soft set denoted by 1_E , where $\underline{1}(x) = 1 \ \forall \ x \in X$.
iii) f_E is a fuzzy soft subset of g_E, denoted by $f_E \sqsubseteq g_E$ if $f(e) \subseteq g(e)$ $\forall \ e \in E$.
iv) f_E and g_E are equal if $f_E \sqsubseteq g_E$ and $g_E \sqsubseteq f_E$. It is denoted by $f_E = g_E$.
v) The complement of f_E is denoted by f_E^c where, $f^c : E \longrightarrow I^X$ is a mapping
defined by $f(e)^c = \underline{1} - f(e)$ for all $e \in E$. Clearly, $(f_E^c)^c = f_E$.
vi) The union of f_E and g_E is a fuzzy soft set h_E defined by, $h(e) = f(e) \cup g(e)$ for all $e \in E$. h_E is denoted by $f_E \sqcup g_E$.
vii) The intersection of f_E and g_E is a fuzzy soft set h_E defined by,

$h(e) = f(e) \cap g(e)$ for all $e \in E$. h_E is denoted by $f_E \sqcap g_E$.

Definition 2.7 [4]
A fuzzy soft set f_E over X is said to be a fuzzy soft point if there is $e \in E$ such that $f(e)$ is a fuzzy point in X (*i.e.* there is $x \in X$ such that $f(e)(x) = \alpha \in (0,1]$, $f(e)(y) = 0$ for all $y \in X - \{x\}$) and $f(e') = \underline{0}$ for every $e' \in E - \{e\}$. It will be denoted by x_α^e. The set of all fuzzy soft points in X is denoted by $FSP(X,E)$.
The fuzzy soft point x_α^e is called belongs to a fuzzy soft set f_E , denoted by $x_\alpha^e \tilde{\in} f_E$ iff $\alpha \leq f(e)(x)$. Every non-null fuzzy soft set f_E can be expressed as the union of all the fuzzy soft points belonging to f_E.

The complement of a fuzzy soft point x_α^e is a fuzzy soft set over X, denoted by $(x_\alpha^e)^c$ and given by, $(x_\alpha^e)^c(e) = \underline{1} - (x_\alpha^e)(e)$ and $(x_\alpha^e)^c(e') = \underline{1}\ \forall\, e' \in E - \{e\}$.

Definition 2.8 [4,20]
Let X be an universe set, E be a fixed set of parameters and δ be the collection of fuzzy soft sets over X satisfies the following conditions:
i) 0_E, 1_E belong to δ,
ii) the union of any number of fuzzy soft sets in δ belongs to δ,
iii) the intersection of any two fuzzy soft sets in δ belongs to δ.

In this case (X,δ,E) is called a fuzzy soft topological space. The members of δ are called fuzzy soft open sets in X and denoted by, $FSO(X,\delta,E)$.
A fuzzy soft set f_E over X is called fuzzy soft closed in X iff $f_E^c \in \delta$ and denoted by $FSC(X,\delta,E)$.

Notation. For $x_\alpha^e \in FSP(X)$ the fuzzy soft set $O_{x_\alpha^e}$ refers to a fuzzy soft open set contains x_α^e and $O_{x_\alpha^e}$ is called a fuzzy soft neighborhood of x_α^e. The fuzzy soft neighborhood system of x_α^e denoted by, $N_E(x_\alpha^e)$ is the family of all its fuzzy soft neighborhoods. In general, for $f_E \in FSS(X,E)$ the notation O_{f_E} refers to a fuzzy soft open set contains f_E and is called a fuzzy soft neighborhood of f_E.

Definition 2.9 [4,20,22]
Let (X, δ, E) be a fuzzy soft topological space and $f_E \in FSS(X, E)$. Then:
i) The fuzzy soft interior of f_E is the fuzzy soft set denoted by f_E° and given by $f_E^\circ = \sqcup \{ g_E : g_E \in \delta \text{ and } g_E \sqsubseteq f_E \}$, that is f_E° is a fuzzy soft open set. Indeed it is the largest fuzzy soft open set contained in f_E. It is clear that $x_\alpha^e \tilde{\in} f_E^\circ$ if and only if there is $O_{x_\alpha^e} \in N_E(x_\alpha^e)$ such that $O_{x_\alpha^e} \sqsubseteq f_E$

ii) The fuzzy soft closure of f_E is the fuzzy soft set denoted by $\overline{f_E}$ and given by $\overline{f_E} = \sqcap \{ g_E : g_E \in \delta^c \text{ and } f_E \sqsubseteq g_E \}$, that is $\overline{f_E}$ is a fuzzy soft closed set, Clearly, $\overline{f_E}$ is the smallest fuzzy soft closed set over X which contains f_E.

3. Some new concepts with some results.

Definition 3.1
Let $A \subseteq X$. The soft characteristic function of A denoted by $\tilde{\chi}_A$ is a fuzzy
soft set $\tilde{\chi}_A : E \longrightarrow I^X$ defined by $\tilde{\chi}_A(e) = \chi_A \ \forall e \in E$, where $\chi_A : X \longrightarrow \{0, 1\}$ is
the characteristic function of A. (*i.e.* $\tilde{\chi}_A$ can be defined as a set of ordered pairs $\tilde{\chi}_A = \{(e, \chi_A) : e \in E , \chi_A \in I^X\}$.

Example 3.2
Let $X = \{x, y, z\}$, $A = \{x, z\} \subset X$ and $E = \{e_1, e_2\}$, then the characteristic function of A is the set,
$\tilde{\chi}_A = \{(e_1, (x_1, y_0, z_1)), (e_2, (x_1, y_0, z_1))\}$ which is a fuzzy soft set on X.

Proposition 3.3
Let X be a universe set, $A, B \subseteq X$ and $\{A_i : i \in J\} \subseteq P(X)$ Then:
i) $\tilde{\chi}_\phi = 0_E$ and $\tilde{\chi}_X = 1_E$,
ii) $A \subseteq B \Longrightarrow \tilde{\chi}_A \sqsubseteq \tilde{\chi}_B$,
iii) $\tilde{\chi}_A \sqcap \tilde{\chi}_B = \tilde{\chi}_{A \cap B}$,
iv) $\sqcup_i \tilde{\chi}_{A_i} = \tilde{\chi}_{\cup_i A_i}$,

v) $\tilde{\chi}_A^c = \tilde{\chi}_{A^c}$.

Proof. It is straightforward verification of the above definition.

Definition 3.4

Let $F_E \in SS(X,E)$. The soft characteristic function of F_E is a fuzzy soft set denoted by $\tilde{\chi}_{F_E}$ and given by a set of ordered pairs $\tilde{\chi}_{F_E} = \{(e, \chi_{F(e)}): e \in E\ ,\ \chi_{F(e)} \in I^X\}$, where $\tilde{\chi}_{F_E}(e) = \chi_{F(e)}\ \ \forall e \in E$ and $\chi_{F(e)}: X \longrightarrow \{0,1\}$.

Example 3.5

Let $X = \{x,y,z\}$, $E = \{e_1,e_2\}$, then $F_E = \{(e_1,\{x,y\}),(e_2,\{x\})\}$ is a soft set over X and the soft characteristic function of F_E is, $\tilde{\chi}_{F_E} = \{(e_1,(x_1,y_1,z_0)),$
$(e_2,(x_1,y_0,z_0))\}$ which is a fuzzy soft set over X.

One can to prove the following proposition by suing the above definition.

Proposition 3.6

Let $F_E, G_E \in FSS(X,E)$ and $\{F_{iE}: i \in J\} \subseteq FSS(X,E)$. Then:

i) $\tilde{\chi}_{\phi_E} = 0_E$ *and* $\tilde{\chi}_{X_E} = 1_E$,

ii) $F_E \tilde{\subseteq} G_E \Longrightarrow \tilde{\chi}_{F_E} \sqsubseteq \tilde{\chi}_{G_E}$,

iii) $\tilde{\chi}_{F_E} \sqcap \tilde{\chi}_{G_E} = \tilde{\chi}_{F_E \tilde{\cap} G_E}$,

iv) $\sqcup_i \tilde{\chi}_{F_{iE}} = \tilde{\chi}_{\tilde{\cup}_i F_{iE}}$,

v) $\tilde{\chi}_{F_E}^c = \tilde{\chi}_{F_E^c}$.

Definition 3.7

Let $f_E \in FSS(X,E)$. Then the soft support of f_E, denoted by $Ssup(f_E)$ is a soft set given by, $Ssup(f_E) = \{(e,\ S(f(e)): e \in E\}$, where $S(f(e))$ is the support of fuzzy set $f(e)$, given by $S(f(e)) = \{x \in X: f(e)(x) > 0\} \subseteq X$.

Example 3.8

Let $X = \{x,y,z\}$, $E = \{e_1,e_2\}$, then $f_E = \{(e_1,(x_{0.3},z_{0.2})),(e_2,y_{0.5})\}$ is a fuzzy

soft set over X and the soft support of
f_E is, $Ssup(f_E) = \{(e_1, \{x, z\}), (e_2, \{y\})\}$ which is a soft set over X.

Proposition 3.9

Let f_E, $g_E \in FSS(X, E)$ and $\{f_{iE} : i \in J\} \subseteq FSS(X, E)$. Then:

i) $Ssup(1_E) = X_E$ *and* $Ssup(0_E) = \emptyset_E$,

ii) $f_E \sqsubseteq g_E \Rightarrow Ssup(f_E) \tilde{\subseteq} Ssup(g_E)$,

iii) $\tilde{\bigcup}_{i \in J} Ssup(f_{iE}) = Ssup(\bigsqcup_{i \in J} f_{iE})$,

iv) $Ssup(f_E) \tilde{\cap} Ssup(g_E) = Ssup(f_E \sqcap g_E)$.

Proof. It is straightforward verification of the above definition.

Definition 3.10

Let $\alpha \in I$. The fuzzy soft constant set over X is a fuzzy soft set denoted by $\underline{\alpha}_E$ and given by, $\underline{\alpha}_E = \{(e, \underline{\alpha}(e)) : e \in E\}$, where $\underline{\alpha}(e)(x) = \alpha \ \forall x \in X$ which is called the constant fuzzy set. A fuzzy soft topological space (X, δ, E) is said to be fully stratified iff $\underline{\alpha}_E \in \delta \ \forall \alpha \in I$.

Definition 3.11 [4]

The fuzzy soft sets f_E and g_E in (X, E) are called fuzzy soft quasi-coincident, denoted by $f_E q g_E$ iff there exist $e \in E$, $x \in X$ such that $f(e)(x) + g(e)(x) > 1$. If f_E is not fuzzy soft quasi-coincident with g_E, then we write $f_E \tilde{q} g_E$, that is $f_E \tilde{q} g_E$ iff $f(e)(x) + g(e)(x) \leq 1$, *i.e.* $f(e)(x) \leq g^c(e)(x)$ for all $x \in X$ and $e \in E$.

A fuzzy soft point x_α^e is said to be soft quasi-coincident with f_E, denoted by $x_\alpha^e q f_E$ iff there exists $e \in E$ such that $\alpha + f(e)(x) > 1$.

Example 3.12

Let $X = \{x, y, z\}$ and $E = \{e_1, e_2\}$, then $f_E = \{(e_1, (x_{0.3}, y_{0.4}, z_{0.2})), (e_2, (x_{0.3}, y_{0.7}))\}$, $g_E = \{(e_1, (x_{0.3}, y_{0.8}, z_1)), (e_2, (x_{0.3}, y_{0.2})\}$ and $h_E = \{e_1, (x_{0.6}, y_{0.4}, z_{0.4})), (e_2, (x_{0.3}, z_{0.7}))\}$ are fuzzy soft sets over X and, $f_E q g_E$, $f_E \tilde{q} h_E$ and $x_{0.5}^{e_1} \tilde{q} f_E$ while $x_{0.9}^{e_2} q g_E$.

Proposition 3.13 [4]

Let x_α^e , $y_\beta^e \in FSP(X,E)$, $f_E, g_E \in FSS(X,E)$ and $\{f_{iE} : i \in J\} \subseteq FSS(X,E)$.
Then:
1) $f_E \tilde{q} g_E \Leftrightarrow f_E \sqsubseteq g_E^c$,
2) $f_E \sqcap g_E = 0_E \Longrightarrow f_E \tilde{q} g_E$,
3) $f_E q g_E \Leftrightarrow x_\alpha^e q g_E$ for some $x_\alpha^e \tilde{\in} f_E$,
5) $x_\alpha^e \tilde{q} f_E \Leftrightarrow x_\alpha^e \tilde{\in} f_E^c$,
6) $f_E \sqsubseteq g_E \Leftrightarrow(x_\alpha^e q f_E \Longrightarrow x_\alpha^e q g_E \; \forall \; x_\alpha^e)$,
7) $x_\alpha^e q(\sqcup_{i \in J} f_{iE}) \Leftrightarrow x_\alpha^e q f_{iE}$ for some f_{iE}.
8) $f_E \tilde{q} f_E^c$.

Now in the next propositions we give more properties for the concept of soft quasi-coincident of fuzzy soft sets.

Proposition 3.14

Let x_α^e , $y_\beta^e \in FSP(X,E)$, $f_E, g_E, h_E \in FSS(X,E)$ and $\{f_{iE} : i \in J\} \subseteq FSS(X,E)$. Then we have:
1) $f_E \tilde{q} g_E$, $h_E \sqsubseteq g_E \Longrightarrow f_E \tilde{q} h_E$,
2) $f_E q g_E$, $g_E \sqsubseteq h_E \Longrightarrow f_E q h_E$,
3) if $x_\alpha^e q(\sqcap_{i \in J} f_{iE})$, then $x_\alpha^e q f_{iE}, \forall \; i \in j$,
4) $f_E \tilde{q} g_E \Leftrightarrow g_E \tilde{q} f_E$,
5) $x \neq y \Longrightarrow x_\alpha^e \tilde{q} y_\beta^e \;\; \forall \; \alpha , \beta \in I$,
6) $x_\alpha^e \tilde{q} y_\beta^e \Leftrightarrow x \neq y \; or \; (x = y \; and \; \alpha + \beta \leq 1)$.

Proof. As a sample we prove the cases 3), 5) and 6).
3) Let $x_\alpha^e q(\sqcap f_{iE})$, then $x_\alpha^e \tilde{\notin} (\sqcap f_{iE})^c \Longrightarrow x_\alpha^e \tilde{\notin} \sqcup f_{iE}^c \Longrightarrow x_\alpha^e \tilde{\notin} f_{iE}^c$ for all $i \in J$,
implies $x_\alpha^e q f_{iE}$ for all $i \in J$.
5) Let $x \neq y$. Suppose $x_\alpha^e q y_\beta^e$, then there exist $e \in E$ and $x, y \in X$ such that $\alpha + \beta >1$, that is must be $x = y$, this is contradiction. Hence $x_\alpha^e \tilde{q} x_\beta^e$.
6) Let $x_\alpha^e \tilde{q} y_\beta^e$. Suppose $x = y$ and $\alpha + \beta > 1$. Take $\alpha = \beta = 1$, then $x_1^e q x_1^e$, this is contradiction. Hence $x \neq y \; or \; (x = y \; and \; \alpha + \beta \leq 1)$.
Conversely, the proof follows from 5) and from Definition 3.11.

Proposition 3.15

Let (X, δ, E) be a fuzzy soft topological space and $x_\alpha^e \in FSP(X, E)$.
Then:
i) $g_E q f_E \Leftrightarrow g_E q \overline{f_E}, \ \forall g_E \in FSO(X, \delta, E)$,
ii) $x_\alpha^e q \overline{f_E} \Leftrightarrow O_{x_\alpha^e} q f_E \ \forall O_{x_\alpha^e} \in N_E(x_\alpha^e)$,
Proof.
i) $g_E \tilde{q} f_E \Leftrightarrow f_E \sqsubseteq g_E^c \Leftrightarrow \overline{f_E} \sqsubseteq \overline{g_E^c} = g_E^c \Leftrightarrow g_E \tilde{q} \overline{f_E}$.
ii) $x_\alpha^e \tilde{q} \overline{f_E} \Leftrightarrow x_\alpha^e \tilde{\in} \overline{f_E}^{\,c} = f_E^{c\circ} \Leftrightarrow$ there exists $O_{x_\alpha^e} \in N_E(x_\alpha^e)$ such that,
$O_{x_\alpha^e} \sqsubseteq f_E^c \Leftrightarrow O_{x_\alpha^e} \tilde{q} f_E$.

4. Some constructed fuzzy soft topologies and relationships.
In this section, we drive some fuzzy soft topologies form other topological structures such as, crisp topology, soft topology, and fuzzy topology and vice versa. Also, we study the relationships between them.

First, we give a simple comparative between crisp set, fuzzy set, soft set, and fuzzy soft sets as in the following remarks and examples.

Proposition 4.1
i) Every crisp set is a fuzzy set [11], but the converse is not necessary true,
ii) Every fuzzy set is a soft set [1], but the converse is not necessary true,
iii) Clearly, if $E \neq \emptyset$, then every crisp set A can be considered as a soft set
in the form $F_E = \{(e, A) : e \in E\ ,\ A \subseteq X\}$, but the converse may not be true.

Example 4.2
Let $X = \{x, y, z\}, E = \{e_1, e_2\}$ and $A = \{x, z\} \subseteq X$, then we can write A as:
i) A fuzzy set in the form $A = (x_1, y_0, z_1)$. But the fuzzy set $B = (x_{02}, y_1, z_{01})$,
is not a crisp set in X.

ii) A soft set over X in the form $F_E = \{(e_1, \{x, z\}), (e_2, \{x, z\})\}$. But the soft set
$G_E = \{(e_1, \{x\}), (e_2, \{x, y\})\}$ is not crisp set and not fuzzy set on X.

Remark 4.3 From Definition 3.1, we observe, every crisp set $A \subseteq X$ can be considered as a fuzzy soft set in the form $f_E = \{(e, \chi_A): e \in E, \chi_A: X \longrightarrow \{0,1\}\}$, but the converse is not true. The following example shows this fact.

Example 4.4
Let $X = \{x, y, z, w\}$, $E = \{e_1, e_2\}$ and $A = \{x, y\} \subseteq X$, then we can write A as a
fuzzy soft set over X in the form $f_E = \{(e_1, (x_1, y_1)), (e_2, (x_1, y_1))\}$. But the fuzzy soft set $g_E = \{(e_1, (x_{0.5}, z_1)), (e_2, (x_{0.2}, y_{0.1}))\}$ is not a crisp set in X.

Remark 4.5 From Definition 3.4, we observe, every soft set F_E on X can be considered as a fuzzy soft set in the form $f_E = \{ (e, \chi_{F_{(e)}}): e \in E, \chi_{F_{(e)}}: X \longrightarrow \{0,1\}\}$, but the converse is not true. The following example shows this fact.

Example 4.6
Let $X = \{x, y, z, w\}$, $E = \{e_1, e_2\}$ and $F_E = \{(e_1, \{x, z\}), (e_2, \{z\}))\}$, then F_E is
a soft set over and we can write it as a fuzzy set over X in the form,
$f_E = \{(e_1, (x_1, z_1)), (e_2, z_1)\}$. But the fuzzy soft set $G_E = \{(e_1, (x_{0.5}, z_1)), (e_2, (x_{0.2}, y_{0.1}))\}$ is not a soft set over X.

Remark 4.7
i) Clearly, if $E \neq \emptyset$, then every fuzzy set $A \in I^X$ can be considered as a
fuzzy soft set, in the form $f_E = \{(e, A): e \in E, A \in I^X\}$, but the converse is
not necessary true. The following example show this fact.
ii) Every parameterized family of fuzzy sets in X is a fuzzy soft set on X [6].

Example 4.8

Let $X = \{x, y, z\}, E = \{e_1, e_2\}$ and $A = (x_{02}, y_1, z_{0.1}) \in I^X$, then we can write A as fuzzy soft set on X in the form $F_E = \{(e_1, (x_{0.2}, y_1, z_{0.1})),$ $(e_2, (x_{0.2}, y_1, z_{0.1}))\}$. But the fuzzy soft set $g_E = \{(e_1, (x_{0.5}, z_1)), (e_2, (x_{0.2}, y_{0.1}))\}$ is not fuzzy set on X.

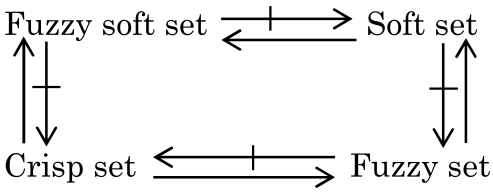

Diagram 1.

Show the relation between crisp set, fuzzy set, soft set and fuzzy soft set

In the next theorems, we show how to generate some fuzzy soft topologies from crisp topology, soft topology, and fuzzy topology and vice versa.

By using Proposition 3.3 one can easily prove the following theorems.

Theorem 4.9

Let (X, τ) be an ordinary topological space. Then the collection, $\delta_\tau = \{ \tilde{\chi}_A : A \in \tau\}$ defines a fuzzy soft topology on X induced by τ.

Theorem 4.10

Every fuzzy soft topological space (X, δ, E) defines a topology on X in the

form $\tau_\delta = \{A \subseteq X : \tilde{\chi}_A \in \delta\}$ which is induced by δ.

Theorem 4.11

Let (X, τ^*, E) be a soft topological space. Then the collections:

i) $\delta_{\tau^*} = \{f_E \in FSS(X, E) : Ssup(f_E) \in \tau^*\}$,

ii) $\delta_\Delta = \{\tilde{\chi}_{F_E} : F_E \in \tau^*\}$,

define the fuzzy soft topologies on X which are induced by τ^*.

Proof. To prove the case *i*) by using Proposition 3.9 we have,
1) $0_E, 1_E \in \delta_{\tau^*}$ because, $sup(0_E) = \emptyset_E \in \tau^*$ and $Ssup(1_E) = X_E \in \tau^*$.
2) Let $f_E, g_E \in \delta_{\tau^*}$, then $Ssup(f_E) \in \tau^*$and $Ssup(g_E) \in \tau^*$. Since, $Ssup(f_E) \tilde{\cap} Ssup(g_E) = Ssup(f_E \sqcap g_E) \in \tau^*$, then $f_E \sqcap g_E \in \delta_{\tau^*}$.
3) Let $\{f_{iE}: i \in J\} \subseteq \delta_{\tau^*}$, then $Ssup(f_{iE}) \in \tau^* \ \forall\, i \in J$. By Proposition 3.9, $Ssup(\sqcup_i f_{iE}) = \tilde{\cup}_i Ssup(f_{iE}) \in \tau^*$, then $\sqcup_i f_{iE} \in \delta_{\tau^*}$. Hence the result holds.
The proof for the case *ii*) follows from Proposition 3.6.

Example 4.12

(1) Let $X = \{x, y, z\}$, $E = \{e_1, e_2\}$, then $\tau^* = \{\emptyset_E, X_E, F_E = \{(e_1, \{x\})\}, G_E = \{(e_2, \{z\})\}, H_E = \{(e_1, \{x\}), (e_2, \{z\})\}\}$ is a soft topology on X and the family, $\delta_{\tau^*} = \{0_E, 1_E, f_E, g_E, h_E\} \cup \{\underline{\alpha}_E: \alpha \in I - \{0,1\}\}$ is a fuzzy soft topology on X, induced by τ^* where $\underline{\alpha_E}(e) = \underline{\alpha} \ \forall e \in E$, $f_E = \{(e_1, (x_\alpha, y_0, z_0)), (e_2, (x_0, y_0, z_0))\}$ $g_E = \{(e_1, (x_0, y_0, z_0)), (e_2, (x_0, y_0, z_\alpha))\}$, $h_E = \{(e_1, (x_\alpha, y_0, z_0)), (e_2, (x_0, y_0, z_\alpha))\}$.

(2) Let $X = \{x, y\}$, $E = \{e_1, e_2\}$, then $\tau^* = \{\emptyset_E, X_E , F_E = \{(e_1, \{x\}), (e_2, \{y\})\}\}$ is a soft topology on X and $\delta_\Delta = \{0_E, 1_E, f_E = \{(e_1, (x_1, y_0)) , (e_2, (x_0, y_1))\}\}$ is a fuzzy soft topology on X which is induced by τ^*.

Theorem 4.13

Let (X, δ, E) be a fuzzy soft topological space, then the collections:
i) $\tau^*_\delta = \{Ssup(f_E) \in SS(X, E) : f_E \in \delta\}$,
ii) $\tau^*_\Delta = \{F_E \in SS(X, E) : \tilde{\chi}_{F_E} \in \delta\}$,
define the soft topologies on X which are induced by δ.

Proof. i) The proof follows by using Proposition 3.9.
ii) Let (X, δ, E) be a fuzzy soft topological, then by using Proposition 3.7 we have: 1) $\emptyset_E, X_E \in \tau^*_\Delta$ because, $\tilde{\chi}_{\emptyset_E} = \tilde{0}_E \in \delta$ and $\tilde{\chi}_{X_E} = \tilde{1}_E \in \delta$.
2) Let $F_E, G_E \in \tau^*_\Delta$, then $\tilde{\chi}_{F_E}$, $\tilde{\chi}_{G_E} \in \delta$. Since $\tilde{\chi}_{F_E \tilde{\cap} G_E} = \tilde{\chi}_{F_E} \sqcap \tilde{\chi}_{G_E} \in \delta$, then $F_E \tilde{\cap} G_E \in \tau^*_\Delta$.
3) Let $\{F_{iE}: i \in J\} \subseteq \tau^*_\Delta$, then $\tilde{\chi}_{F_{iE}} \in \delta \ \forall\, i \in J$. Since $\tilde{\chi}_{\tilde{\cup}_i F_{iE}} = \sqcup_i \tilde{\chi}_{F_{iE}} \in \delta$, then
$\tilde{\cup}_i F_{iE} \in \tau^*_\Delta$. Hence we obtain the result.

Example 4.14

(1) Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ and $\delta = \{ 0_E, 1_E, f_E, g_E\}$ be a fuzzy soft topology

on X, where $f_E = \{(e_1, (x_{0.3}, y_0, z_0)), (e_2, (x_{0.5}, y_{0.7}, z_0))\}$, $g_E = \{(e_1, (x_{0.2}, y_0, z_0))\}$. Then $\tau_\delta^* = \{\emptyset_E, X_E, F_E = \{(e_1, \{x\}), (e_2, \{x, y\}),\ G_E = \{(e_1, \{x\})\}\}$ is a soft topology on X which is induced by δ.

(2) Let $X = \{x, y, z\}$, $E = \{e_1, e_2\}$, then $\delta = \{0_E, 1_E, f_E, g_E\}$ is a fuzzy soft topology on X where, $f_E = \{(e_1, (x_1, y_0, z_{0.5})), (e_2, (x_o, y_1, z_0))\}$, $g_E = \{(e_1, (x_1, y_0, z_0)), (e_2, (x_0, y_1, z_0))\}$ and
$\tau_\Delta^* = \{\emptyset_E, X_E, G_E = \{(e_1, \{x\}), (e_2, \{y\})\}\}$
is a soft topology on X which is induced by δ.

Proposition 4.15

Let (X, τ) be a topological space, (X, τ^*, E) be a soft topological space and (X, δ, E) be a fuzzy soft topological space. Then:
i) $\delta \subseteq \delta_{\tau_\delta^*}$ and $\tau^*_{\delta_{\tau^*}} = \tau^*$,
ii) $\delta_{\tau^*_{\delta_{\tau^*}}} = \delta_{\tau^*}$,
iii) $\underline{\alpha}_E \in \delta_{\tau^*}$ for all $\alpha \in I$,
iv) $F_E \in \tau^* \Longrightarrow \tilde{\chi}_{F_E} \in \delta_{\tau^*}$, in particular $\delta_\Delta \subseteq \delta_{\tau^*}$.
Proof. Obvious

Theorem 4.16

i) Let $(X, \hat{\tau})$ be a fuzzy topological space and E is a parameter set, then the
collection $\delta_{\hat{\tau}} = \{f_E : f(e) = A\ \ \forall\, e \in E$ and $A \in \hat{\tau}\ \}$ defines a fuzzy soft topology on X induced by $\hat{\tau}$.
ii) Let (X, δ, E) be a fuzzy soft topological space, then the collection, $\hat{\tau}_\delta^e = \{f(e) : f_E \in \delta\}$, for every $e \in E$ defines a fuzzy topology on X [18].
Proof.
i) Let $(X, \hat{\tau})$ be a fuzzy topological space and E is a parameter set, then:
1) Since $\underline{0}$, $\underline{1} \in \hat{\tau}$, $f(e) = \underline{0}$ *and* $f(e) = \underline{1}\ \forall\, e \in E$ then, $0_E, 1_E \in \delta_{\hat{\tau}}$.

2) Let $f_E, g_E \in \delta_{\hat{\tau}}$, then there are $A, B \in \hat{\tau}$ such that $f(e) = A$, $g(e) = B$ $\forall e \in E$. Since $f(e) \cap g(e) = (f \cap g)(e) = A \cap B$ $\forall e \in E$ and $A \cap B \in \hat{\tau}$, then $f_E \sqcap g_E \in \delta_{\hat{\tau}}$.

3) Let $\{f_{iE}: i \in J\} \subseteq \delta_{\hat{\tau}}$, then there are $\{A_i: i \in J\} \subseteq \hat{\tau}$ such that $f_i(e) = A_i$ $\forall e \in E$

,$i \in J$. Since $\cup f_i(e) = (\cup f_i)(e) = \cup A_i$ $\forall e \in E$, $i \in J$ and $\cup A_i \in \hat{\tau}$, then $\sqcup f_{iE} \in \delta_{\hat{\tau}}$.

Hence the result holds.

Theorem 4.17

i) Let (X, τ) be a topological space and E is a parameter set, then the collection $\tau_\tau^* = \{F_E: F(e) = A \ \forall e \in E \text{ and } A \in \tau\}$ defines a soft topology on X induced by τ [9].

ii) Let (X, τ^*, E) be a soft topological space, then the collection, $\tau_{\tau^*}^e = \{F(e) : F_E \in \tau^*\}$, for every $e \in E$ defines a topology on X [21].

Example 4.18

Let $X = \{x, y\}$, $E = \{e_1, e_2\}$. Then the collections:

i) $\delta = \{0_E, 1_E, f_E = \{(e_1, (x_{0.3}, y_0, z_0)), (e_2, (x_{0.5}, y_{0.7}, z_0))\}\}$ is fuzzy soft topology

on X, but is not topology, not fuzzy topology, and not soft topology.

ii) $\tau^* = \{\emptyset_E, X_E, F_E = \{(e_1, \{x\}), (e_2, X)\}\}$ is soft topology on X, but it's neither topology nor fuzzy topology on X. Clearly, the family $\hat{\tau} = \{\underline{0}$, $\underline{1}$, $A = (x_{0.2}, y_{0.5})\}$ is a fuzzy topology on X, but is not a crisp topology on X.

Corollary 4.19

From the above theorems , remarks, and examples, the implications in the following diagram are hold.

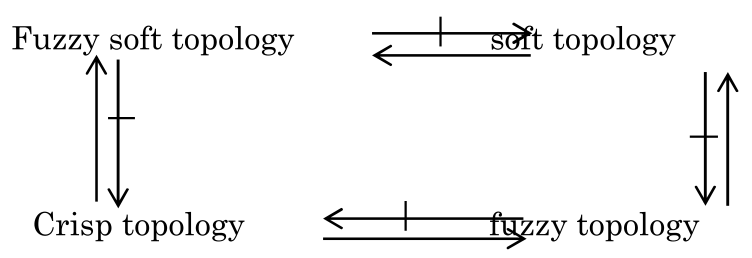

Diagram 2.

Show the relation between topology, fuzzy topology, soft topology and fuzzy soft topology.

5. Conclusion

In this paper, we show how some fuzzy soft topologies are derived from other topological structures and vice versa, as well as the relations between them are showed. The notions of soft characteristic function of crisp and soft set on X, soft support of fuzzy soft sets are given, according to these notions some new results are studied. To extend this work, we could study some properties of fuzzy soft topological spaces in different topological aspects.

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